Highly connected subgraphs of graphs with given independence number

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Abstract

Let G be a graph on n vertices with independence number α. We prove that if n is sufficiently large ($n \geq \alpha^2 k + 1$ will do), then G always contains a k-connected subgraph on at least n/α vertices. The value of n/α is sharp, since G might be the disjoint union of α equally-sized cliques. For $k \geq 3$ and $\alpha = 2, 3$, we shall prove that the same result holds for $n \geq 4(k-1)$ and $n \geq \frac{27(k-1)}{4}$ respectively, and that these lower bounds on n are sharp.

1 Introduction

When can we find a large highly connected subgraph of a given graph G? A classical theorem due to Mader [10] (see also [6]) states that if G has average degree at least $4k$, then G contains a k-connected subgraph H. Mader's theorem does not give a lower bound on the order of H. If G is dense (for instance if $\delta(G)$, the minimum degree of G, is bounded below), it is natural to expect that G in fact contains a *large* highly connected subgraph. A result of Bohman et al. [1] implies that for every graph G of order n with $\delta(G) \geq 4\sqrt{kn}$, the vertex set $V(G)$ admits a partition such that every part induces a k-connected subgraph of order at least $\sqrt{kn}/2$. In a similar direction, by a recent result of Borozan et al. [4], we know that every graph G of order n with $\delta(G) \geq \sqrt{c(k-1)n}$ contains a k-connected subgraph of order at least $\sqrt{(k-1)n/c}$, where $c = 2123/180$. What if we are interested in finding a larger k-connected subgraph, say of order cn? Along these lines, Bollobás and Gyárfás [3] conjectured that for any graph G of order $n \geq 4k-3$, G or its complement $\overline{G}$ contains a k-connected subgraph H of order at least $n - 2(k-1)$. Since either G or $\overline{G}$ is a dense graph, we might expect to find a very large highly connected subgraph in one of them. This

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conjecture was settled affirmatively for $n \geq 13k - 15$ by Liu, Morris and Prince [9], and then for $n > 6.5(k-1)$ by Fujita and Magnant [8].

Suppose next that $\delta(G) \geq cn$. Can we find a k-connected subgraph of G on at least cn vertices? It turns out that the answer is "yes" for sufficiently large $n \geq n_0(c,k)$, and in fact a simple argument gives even more – there exists such a subgraph on at least n/m vertices, where $m = \lfloor 1/c \rfloor$. For instance, if $c > 1/2$, then G itself is k-connected. To see the assertion, suppose that G itself is not k-connected. Then G can be split into two "large" pieces with a separating set of size at most $k-1$. Both pieces must have order at least $cn - (k-1) + 1$, so as not to violate the minimum degree condition. If one of the pieces does not induce a k-connected subgraph, then we split the subgraph, to obtain another two large pieces and a separating set of size at most $k-1$. Now, each of the three large pieces has order at least $cn - 2(k-1) + 1$. We repeat this procedure, as long as one of the existing large pieces does not induce a k-connected subgraph. After t steps, we have $t+1$ large pieces, each with order at least $cn - t(k-1) + 1$. If $m \leq t = O_n(1)$, then there is a large piece with at most $\frac{n}{m+1} \ll cn$ vertices, which is a contradiction. Thus, the procedure must terminate after $t \leq m-1$ steps, giving us $t+1 \leq m$ large pieces, say $X_1, \ldots, X_t$. Finally, let C be the set of accumulated separating vertices, so that $|C| \leq t(k-1)$. We can "redistribute" the vertices of C to the X_i. The minimum degree condition on G implies that every vertex of C must have at least k neighbours in one of the X_i. Therefore, we can write $C = C_1 \dot\cup \cdots \dot\cup C_t$ such that, every vertex of C_i has at least k neighbours in X_i, for every $1 \leq i \leq t$. So we are done, since some set $X_i \cup C_i$ induces a k-connected subgraph on at least $n/m \geq cn$ vertices. Furthermore, the value of n/m for the order of the largest k-connected subgraph is sharp, if $k \geq 4$. To see this, we can let G be the union of m cliques, each with $\lceil n/m \rceil + 1$ vertices, and ordered linearly in such a way that any two successive cliques intersect in at most three vertices, with non-consecutive cliques being disjoint.

Here, we instead focus on another graph parameter which forces G to be dense, but which does not immediately yield a trivial bound for our problem. Such a parameter is the independence number $\alpha(G)$. If a graph G has independence number α, then its complement $\overline{G}$ has clique number α, so that, by Turán's theorem, $\overline{G}$ has average degree at most around $(1 - 1/\alpha)n$, and so G has average degree at least around n/α. It is natural to conjecture that this average degree condition automatically implies that G has a k-connected subgraph on at least n/α vertices. However, this conjecture is false. Indeed, for the cases $\alpha = 2$ and $\alpha = 3$, our graphs in Constructions 5 and 7 (see Section 3) have average degrees $(19/32)n$ and $(307/729)n$, and no k-connected subgraphs of orders $n/2$ and $n/3$ respectively.

Structures in graphs with fixed independence number are widely studied. In particular, the problem of finding a large subgraph with certain properties in a graph with fixed independence number has received much attention. For example, a famous theorem due to Chvátal and Erdős [5] from 1972 states that any graph G on at least three vertices, whose independence number $\alpha(G)$ is at most its connectivity $\kappa(G)$, contains a Hamiltonian cycle. Motivated by this, Fouquet and Jolivet [7] conjectured in 1976 that if, instead, G is a k-connected graph of order n with $\alpha(G) = \alpha \geq k$, then G has a cycle with length at least $\frac{k(n+\alpha-k)}{\alpha}$. Recently, this long standing conjecture was settled affirmatively by O et al. [11].

In this paper, we consider the following question. Fix $k \geq 1$, and let G be a graph on n vertices with independence number α. Can we always find a large k-connected subgraph of G? A little thought shows that, if $n \leq \alpha k$, then there might be no such subgraph, and if

$n \geq \alpha k + 1$, then we are only guaranteed a k-connected subgraph of order $\lceil n/\alpha \rceil$, since in both cases G might consist of the disjoint union of α cliques, each with either $\lceil n/\alpha \rceil$ or $\lfloor n/\alpha \rfloor$ vertices. Such a graph G has the fewest edges among all graphs of independence number α, so it seems that it should be extremal for our problem as well.

In fact, for large n, this construction (which we will call the *disjoint clique construction*, or just DCC) is indeed extremal. Specifically, we prove in Theorem 2 that any graph G of order $n \geq \alpha^2 k + 1$ and independence number α must have a k-connected subgraph of order at least $\lceil n/\alpha \rceil$. However, for smaller values of n, this no longer applies. For instance, when $\alpha = 2$ and $k \geq 3$, there is a graph of order $n = 4k - 5$ and independence number 2 with no k-connected subgraph of order at least $\lceil n/2 \rceil$ (see Construction 5). Also, when $\alpha = 3$ and $k \geq 3$, there is a graph of order $n = \lceil \frac{27(k-1)}{4} \rceil - 1$ (for k odd) or $n = \lceil \frac{27(k-1)}{4} \rceil - 2$ (for k even), and independence number 3, with no k-connected subgraph of order at least $\lceil n/3 \rceil$ (see Construction 7). These examples, however, *are* extremal, in the sense that for any graph with order $n \geq 4k - 4$ (resp. $n \geq \frac{27(k-1)}{4}$ for k odd, and $n \geq \frac{27(k-1)}{4} - 1$ for k even), there is always a k-connected subgraph on at least $\lceil n/2 \rceil$ (resp. $\lceil n/3 \rceil$) vertices, and the DCC is thus optimal. We prove this in Theorem 4 and Theorem 6. In view of the complexity of both Construction 7 and the proof of Theorem 6, we suspect that the generalisation to higher values of α is far from simple.

Although Theorem 6 might appear rather modest, the structural difference of Constructions 5 and 7 shows the difficultly of finding the exact lower bound on n for $\alpha \geq 4$. Indeed, we have no conjecture as to what this lower bound might be. By contrast, the sharpness constructions for both the Bollobás-Gyárfás conjecture and the Fouquet-Jolivet conjecture (now the theorem of O et al.) both consist of one essentially unique example, suggesting that our problem is very different from these others. Therefore, rather than pursuing the correct lower bound on n for all α, we present in Theorem 2 an argument yielding the lower bound of $n \geq \alpha^2 k + 1$ for all $\alpha, k \geq 1$.

For a graph G and sets $X, Y \subset V(G)$, let $G[X]$ denote the subgraph of G induced by X. Let $E(X, Y)$ denote the set of edges between X and Y, and $e(X, Y) = |E(X, Y)|$. For a vertex $v \in V(G)$, let $\Gamma(v)$ denote the set of neighbours of v. We say that v *dominates* X if v is adjacent to every vertex of $X \setminus \{v\}$. For any other terms not defined here, we refer the reader to [2].

2 The case of general α

From now on we fix $k \geq 1$ and $\alpha \geq 1$. Our first observation is that the case $k = 1$ in our problem is trivial. Indeed, if G is a graph of order n and independence number α, then the largest connected component of G must contain at least $\lceil n/\alpha \rceil$ vertices. The case $k = 2$ is a little harder, but is covered by the following result.

Proposition 1. *Let G be a graph of order $n > 2\alpha$ and independence number α. Then G contains a 2-connected subgraph of order at least $\lceil n/\alpha \rceil$.*

Proof. We apply induction on α, with the case $\alpha = 1$ being trivial. Let $\alpha \geq 2$, and G be as in the statement of the proposition. We are done if G is 2-connected, so assume otherwise. We consider the blocks of G, for which we refer the reader to [2] (Ch. III.2) for further information. Recall that a *block* of G is a subgraph which is either an isolated vertex, or a

bridge (with its end-vertices), or a maximal 2-connected subgraph (i.e., not strictly contained in another 2-connected subgraph). An *end-block* of G is a block which is incident with at most one cut-vertex of G, and G must contain an end-block. Note that, since G is not 2-connected, G itself is not an end-block. This means that there exists an end-block $B \subset G$ such that $1 \leq |B| < |G|$ and $\alpha(G - B) = \beta$, where $1 \leq \beta \leq \alpha - 1$.

Now, if $|B| > 2$ then B is 2-connected, so we may assume that $|B| < n/\alpha$. This implies $|G - B| > n - n/\alpha > 2(\alpha - 1) \geq 2\beta$. Applying the induction hypothesis to $G - B$, we see that $G - B$ contains a 2-connected subgraph of order at least $\frac{n-|B|}{\beta} > \frac{n-n/\alpha}{\alpha-1} = n/\alpha$.

Similarly, if $|B| \in \{1, 2\}$, then $G - B$ contains a 2-connected subgraph of order at least $\frac{n-|B|}{\beta} \geq \frac{n-2}{\alpha-1} > n/\alpha$. □

The example of a path on 2α vertices shows that the hypothesis cannot be weakened.

Consequently, we may, if necessary, restrict our attention to the case $k \geq 3$. Our first main result shows that, for large values of n, the disjoint clique construction (DCC) is optimal.

Theorem 2. *Let $\alpha, k \geq 1$ and let G be a graph of order $n \geq \alpha^2 k + 1$ and independence number α. Then G contains a k-connected subgraph of order at least $\lceil n/\alpha \rceil$.*

Proof. Let G have order $n \geq \alpha^2 k + 1$ and independence number α. We wish to find a large k-connected subgraph of G. To this end, we apply induction on α, and assume that the theorem is true for smaller values of α. It is certainly true when $\alpha = 1$, since a clique with at least $k + 1$ vertices is k-connected – note, however, that this is the only place where we use the "+1" in the lower bound on n. Thus, let $\alpha \geq 2$.

Before proceeding with the formal proof, we explain the general strategy. If G itself is not k-connected, there must exist a separating set C, splitting the rest of G into two pieces X and Y. We may then use induction to argue that the independence numbers $\alpha(G[X])$ and $\alpha(G[Y])$ add up to α, and $|X|$ and $|Y|$ are roughly proportional to $\alpha(G[X])$ and $\alpha(G[Y])$ respectively. We keep splitting up $X, Y, \ldots$, assuming each is not k-connected, until we are left with α cliques, each of which has order at least $k + 1$. (This is one place where we need the bound on n, so that the union of the accumulated separating sets C' should not deplete the remaining graph too much.) Finally, we "redistribute" the vertices from C' to the cliques, noting that each $v \in C'$ must dominate some clique, or else the independence number is too high. One such "enhanced" clique must have at least $\lceil n/\alpha \rceil$ vertices, and this will be our desired k-connected subgraph.

Now we begin the formal proof. We consider the largest $i \geq 0$ such that there exists a disjoint union $V(G) = A_0 \dot\cup A_1 \dot\cup \cdots \dot\cup A_i \dot\cup C_i$ with the following four properties:

(i) $A_j \neq \emptyset$ for all $0 \leq j \leq i$ (but we may have $C_i = \emptyset$)

(ii) $E(A_j, A_\ell) = \emptyset$ for all $0 \leq j < \ell \leq i$

(iii) $\alpha(G[A_0]) + \cdots + \alpha(G[A_i]) = \alpha$

(iv) $\frac{\alpha(G[A_j])n}{\alpha} - i(k-1) \leq |A_j| \leq \frac{\alpha(G[A_j])n}{\alpha}$ for all $0 \leq j \leq i$.

That there is such an i follows from setting $i = 0$, $A_1 = V(G)$ and $C_0 = \emptyset$. By (i), G has an independent set of size $i + 1$, and so $i \leq \alpha - 1$. Now, if we can take $i = \alpha - 1$, we are done. To see this, note that in this case properties (i) and (iii) would imply that

$\alpha(G[A_0]) = \cdots = \alpha(G[A_{\alpha-1}]) = 1$, so that $G[A_0], \ldots, G[A_{\alpha-1}]$ are all cliques. Property (iv) then implies that each clique has order at least

$$\frac{n}{\alpha} - (\alpha-1)(k-1) > \alpha k - (\alpha-1)(k-1) = \alpha + k - 1 \geq k+1,$$

and hence each clique is k-connected. Next, every vertex v in $C_{\alpha-1}$ must be adjacent to every vertex in some A_j, or else we would have an independent set with $\alpha+1$ vertices. Hence there is a disjoint union $C_{\alpha-1} = D_0 \dot\cup \cdots \dot\cup D_{\alpha-1}$ such that, for every $0 \leq j \leq \alpha-1$ and $v \in D_j$, v dominates A_j. (Note that any of $C_{\alpha-1}, D_0, \ldots, D_{\alpha-1}$ may be empty.) Now, each $G[A_j \cup D_j]$ is a k-connected subgraph, and one of them must have at least $\lceil n/\alpha \rceil$ vertices, as desired.

Now suppose that G does not have the required k-connected subgraph. Then, from the above argument, the maximum i for which there is a disjoint union $V(G) = A_0 \dot\cup \cdots \dot\cup A_i \dot\cup C_i$ satisfying (i) to (iv) must satisfy $i < \alpha-1$. Fix this i. From property (iii) and the pigeonhole principle, we may assume, relabelling if necessary, that $\alpha(G[A_i]) \geq 2$. From property (iv), we now have

$$\begin{aligned}
|A_i| &\geq \frac{\alpha(G[A_i])n}{\alpha} - i(k-1) \\
&\geq \frac{n}{\alpha} + \frac{n}{\alpha} - (\alpha-2)(k-1) \\
&> \frac{n}{\alpha} + \alpha k - (\alpha-2)(k-1) \\
&= \frac{n}{\alpha} + 2(k-1) + \alpha > \left\lceil \frac{n}{\alpha} \right\rceil.
\end{aligned}$$

Hence $G[A_i]$ is not k-connected, and we have $A_i = X \dot\cup Y \dot\cup C$ such that $X, Y \neq \emptyset$, $|C| \leq k-1$, and $E(X,Y) = \emptyset$. Set $A'_j = A_j$ for $0 \leq j \leq i-1$, $A'_i = X$, $A'_{i+1} = Y$, and $C_{i+1} = C_i \cup C$, so that $V(G) = A'_0 \dot\cup \cdots \dot\cup A'_{i+1} \dot\cup C_{i+1}$. We claim that this disjoint union satisfies properties (i) to (iv) with $i+1$ in place of i, which will contradict the maximality of i. It is clear that (i) and (ii) both hold for $i+1$.

Before we prove (iii) and (iv), we first note that:

$$\text{Both } |X| \leq \frac{\alpha(G[X])n}{\alpha} \text{ and } |Y| \leq \frac{\alpha(G[Y])n}{\alpha} \text{ must hold.} \tag{1}$$

Otherwise, assume that $|X| > \frac{\alpha(G[X])n}{\alpha}$ (without loss of generality). Then

$$|X| > \frac{\alpha(G[X])n}{\alpha} > \alpha(G[X]) \cdot \alpha k \geq \alpha(G[X])^2 k + 1,$$

so that, by induction, G contains a k-connected subgraph on at least $\lceil \frac{|X|}{\alpha(G[X])} \rceil \geq \lceil n/\alpha \rceil$ vertices, a contradiction.

Now we prove (iii) for $i+1$. We have to prove that $\alpha(G[X]) + \alpha(G[Y]) = \alpha(G[A_i])$. Clearly, if $\alpha(G[X]) + \alpha(G[Y]) > \alpha(G[A_i])$, then we can find an independent set in $G[A_i]$ with more than $\alpha(G[A_i])$ vertices, a contradiction. On the other hand, if $\alpha(G[X]) + \alpha(G[Y]) < \alpha(G[A_i])$,

then by property (iv) for i, we have

$$\begin{aligned}
|X| + |Y| &= |A_i| - |C| \\
&\geq \frac{\alpha(G[A_i])n}{\alpha} - i(k-1) - |C| \\
&\geq \frac{\alpha(G[X])n}{\alpha} + \frac{\alpha(G[Y])n}{\alpha} + \frac{n}{\alpha} - i(k-1) - |C| \\
&> \frac{\alpha(G[X])n}{\alpha} + \frac{\alpha(G[Y])n}{\alpha} + \alpha k - (i+1)(k-1) \\
&\geq \frac{\alpha(G[X])n}{\alpha} + \frac{\alpha(G[Y])n}{\alpha} + \alpha k - (\alpha-1)(k-1) \\
&= \frac{\alpha(G[X])n}{\alpha} + \frac{\alpha(G[Y])n}{\alpha} + \alpha + k - 1 \\
&> \frac{\alpha(G[X])n}{\alpha} + \frac{\alpha(G[Y])n}{\alpha},
\end{aligned}$$

which means that either $|X| > \frac{\alpha(G[X])n}{\alpha}$ or $|Y| > \frac{\alpha(G[Y])n}{\alpha}$. This contradicts (1).

Finally, we prove (iv) for $i+1$. Clearly the property holds for A'_j when $0 \leq j \leq i-1$. It remains and suffices to prove that

$$\frac{\alpha(G[X])n}{\alpha} - (i+1)(k-1) \leq |X| \leq \frac{\alpha(G[X])n}{\alpha}, \tag{2}$$

since the analogous inequalities for Y can be proved similarly. Now, the upper bound of (2) follows from (1), so it remains to prove the lower bound. Suppose that $|X| < \frac{\alpha(G[X])n}{\alpha} - (i+1)(k-1)$. Then, since $\alpha(G[X]) + \alpha(G[Y]) = \alpha(G[A_i])$, we must have

$$\begin{aligned}
|Y| &= |A_i| - |X| - |C| \\
&\geq \frac{\alpha(G[A_i])n}{\alpha} - i(k-1) - |X| - |C| \\
&> \frac{\alpha(G[A_i])n}{\alpha} - i(k-1) - \Big(\frac{\alpha(G[X])n}{\alpha} - (i+1)(k-1)\Big) - (k-1) \\
&= \frac{\alpha(G[Y])n}{\alpha},
\end{aligned}$$

which contradicts (1).

It follows that properties (i) to (iv) hold for $i+1$, which is the required contradiction. The induction step for α is now complete, and the theorem is proved. □

One might hope that the condition on n can be relaxed all the way down to $n \geq \alpha k + 1$, in order for the conclusion of Theorem 2 to hold. However, this is not possible. As we have mentioned in the introduction, when $\alpha = 2, 3$, we would require $n \geq 4(k-1)$ and (approximately) $n \geq \frac{27(k-1)}{4}$, respectively. In the following construction, we will see that, for every $\alpha \geq 4$ and $k \geq 3$, there exists a graph $G_{\alpha,k}$ with approximately $\frac{5\alpha(k-1)}{2}$ vertices and independence number α, for which the conclusion of Theorem 2 does not hold. This construction is illustrated in Figure 1.

Construction 3. *Let $k \geq 3$. We construct the graph $G_{4,k}$ as follows. For $1 \leq i \leq 4$, let A_i, B_i, C_i be sets of vertices, where*

$$|A_i| = |B_1| = |B_2| = k-1, \quad |B_3| = |B_4| = k-2,$$
$$|C_1| = |C_2| = \left\lfloor \frac{k-3}{2} \right\rfloor, \quad |C_3| = |C_4| = \left\lfloor \frac{k-1}{2} \right\rfloor,$$

Let v be another vertex. For all i, we add edges to make $G_{4,k}[A_i \cup B_i \cup C_i]$ a clique, and add all edges between the following 10 pairs:

$$(v, A_i), (B_1, B_3), (B_2, B_4), (C_1, C_2), (C_3, C_4), (C_1, C_4), (C_2, C_3).$$

Now let $\alpha \geq 5$. We construct the graph $G_{\alpha,k}$ from $G_{4,k}$ as follows. For $1 \leq j \leq \alpha - 4$, let D_j, E_j be additional sets of vertices, where

$$|D_j| = k-2, \quad |E_j| = \left\lfloor \frac{3(k-1)}{2} \right\rfloor.$$

For all j, we add edges so that $G_{\alpha,k}[\{v\} \cup D_j]$ and $G_{\alpha,k}[D_j \cup E_j]$ are cliques.

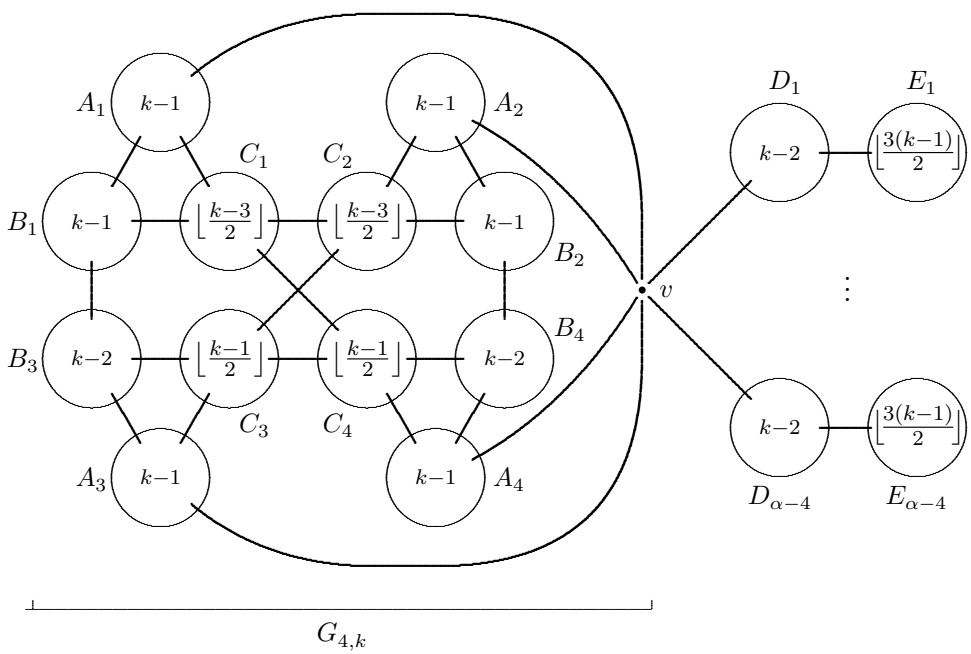

Figure 1. A graph $G_{\alpha,k}$ with $\alpha(G_{\alpha,k}) = \alpha$, and no k-connected subgraph on at least $\lceil n/\alpha \rceil$ vertices.

We remark that in Figure 1, as well as in subsequent figures, a circle indicates a clique, and a line connecting two sets means that all edges between the sets are present.

For every $\alpha \geq 4$ and $k \geq 3$, it is easy to see that $\alpha(G_{\alpha,k}) = \alpha$. This is true even for $k = 3, 4$, where we have $C_1 = C_2 = \emptyset$. Also,

$$|G_{\alpha,k}| = n = \begin{cases} \dfrac{5\alpha(k-1)}{2} - (\alpha - 1) & \text{if } k \text{ is odd,} \\ \dfrac{5\alpha(k-1) - \alpha}{2} - (\alpha - 1) & \text{if } k \text{ is even,} \end{cases}$$

so that

$$\left\lceil \frac{n}{\alpha} \right\rceil = \begin{cases} \dfrac{5(k-1)}{2} & \text{if } k \text{ is odd,} \\ \dfrac{5(k-1)-1}{2} & \text{if } k \text{ is even.} \end{cases}$$

Now, the sets $\{v\}$ and D_j, for $1 \le j \le \alpha - 4$, are all cut-sets of size at most $k-1$ for $G_{\alpha,k}$. This means that any k-connected subgraph of $G_{\alpha,k}$ must be within either $G_{4,k}$, or some $D_j \cup E_j$. Next, $\{v\} \cup C_1 \cup C_3$ is a cut-set of size at most $k-1$ for $G_{4,k}$, so that any k-connected subgraph of $G_{4,k}$ must be within either $H = G_{4,k}[\{v\} \cup \bigcup_{i=1,3} A_i \cup B_i \cup C_i]$ or $H' = G_{4,k}[\{v\} \cup \bigcup_{i=2,4} A_i \cup B_i \cup \bigcup_{\ell=1}^{4} C_\ell]$. Since $C_2 \cup C_4$ is a cut-set of size at most $k-1$ for H', any k-connected subgraph of H' must be within $H'' = G_{4,k}[\{v\} \cup \bigcup_{i=2,4} A_i \cup B_i \cup C_i]$, since $G_{4,k}[\bigcup_{i=1}^{4} C_i]$ does not contain a k-connected subgraph. Finally, $\{v\} \cup B_3$ and $\{v\} \cup B_4$ are cut-sets of size $k-1$ for H and H'', respectively. It follows that a k-connected subgraph of $G_{\alpha,k}$ with the largest order must be either $G_{\alpha,k}[A_i \cup B_i \cup C_i]$ for some $1 \le i \le 4$, or $G_{\alpha,k}[D_j \cup E_j]$ for some $1 \le j \le \alpha - 4$. It is easy to check that each of these subgraphs has order $\lceil n/\alpha \rceil - 1$.

3 The cases $\alpha = 2$ and $\alpha = 3$

In this section, we consider our problem for the cases $\alpha = 2$ and $\alpha = 3$. While we believe that the bound $n \ge \alpha^2 k + 1$ in Theorem 2 is far from being sharp, we will obtain the sharp results for $\alpha = 2, 3$. When $\alpha = 2$, the best bound on n improves that of Theorem 2 slightly.

Theorem 4. *Let $k \ge 3$, and let G be a graph of order $n \ge 4(k-1)$ and independence number 2. Then G has a k-connected subgraph of order at least $\lceil n/2 \rceil$.*

Proof. Let k, n and G be as in the statement of the theorem. If G is k-connected we are done, so assume that G is not k-connected. Then we can write $V(G) = A_1 \dot\cup A_2 \dot\cup C$ with $|C| \le k-1$, $0 < |A_1| \le |A_2|$ and $E(A_1, A_2) = \emptyset$. Since $\alpha(G) = 2$, each $G[A_i]$ must be complete, and moreover we can write $C = C_1 \dot\cup C_2$ such that each vertex in C_i dominates all of A_i.

If $|A_1| \ge k$, then also $|A_2| \ge k$. The larger of $G[A_1 \cup C_1]$ and $G[A_2 \cup C_2]$, with at least $n/2 > k$ vertices, is k-connected, and hence is our desired subgraph.

If instead $|A_1| \le k-1$, then $|A_2| = n - |C| - |A_1| \ge n - 2(k-1) \ge n/2 > k$, so $G[A_2]$ is our desired subgraph. □

The bound on n in Theorem 4 is indeed best possible, as can be seen by the following construction, illustrated in Figure 2.

Construction 5. *Let $k \ge 3$. Let G be formed from five vertex sets A, B, C, D and E, of orders $k-1, \lfloor \frac{k-1}{2} \rfloor, \lceil \frac{k-1}{2} \rceil, k-1$ and $k-2$ respectively, and with all edges within each set, and from A to B, B to D, D to E, E to C and C to A.*

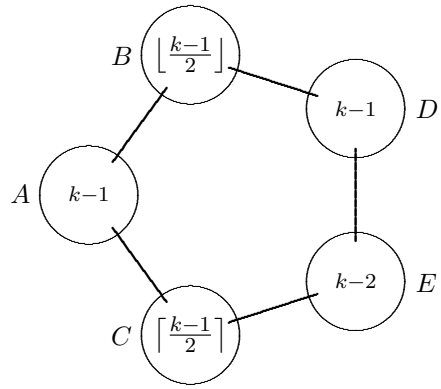

Figure 2. A graph G with $\alpha(G) = 2$, and no k-connected subgraph on at least $\lceil n/2 \rceil$ vertices.

We can easily see that $|G| = 4k - 5 = n$, $\alpha(G) = 2$, and the largest k-connected subgraph in G is $G[D \cup E]$, of order $2k - 3 = \lceil n/2 \rceil - 1$.

Now, we consider $\alpha = 3$. We will prove the following theorem, which is our second main result of this paper.

Theorem 6. *Let $k \geq 3$, and let G be a graph with order n, where*

$$n \geq \begin{cases} \dfrac{27(k-1)}{4} & \text{if } k \text{ is odd,} \\ \dfrac{27(k-1)}{4} - 1 & \text{if } k \text{ is even,} \end{cases} \tag{3}$$

and independence number 3.

Then G has a k-connected subgraph of order at least $\lceil n/3 \rceil$.

Rather surprisingly, the lower bound for n as stated in (3) is the best possible, so that in Theorem 6, the DCC is once again optimal. For smaller values of n, we have the following construction, illustrated in Figure 3.

Construction 7. *Let $k \geq 3$. We construct G as follows. Let $A, B, C_1, C_2, D_1, D_2, E_1, E_2$ and F be nine vertex sets, where*

$$|A| = |D_1| = |D_2| = |E_1| = |E_2| = k - 1,$$
$$|B| = \left\lceil \frac{k+1}{4} \right\rceil, |C_1| = \left\lfloor \frac{k-1}{2} \right\rfloor, |C_2| = \left\lfloor \frac{3k-5}{4} \right\rfloor, |F| = \left\lfloor \frac{k-3}{4} \right\rfloor.$$

We add all edges within each set, and between the following 14 pairs (where $i = 1, 2$):

$$(A, C_i), (C_1, C_2), (C_i, D_i), (D_i, E_i), (D_i, F), (E_i, F), (B, E_i), (A, B).$$

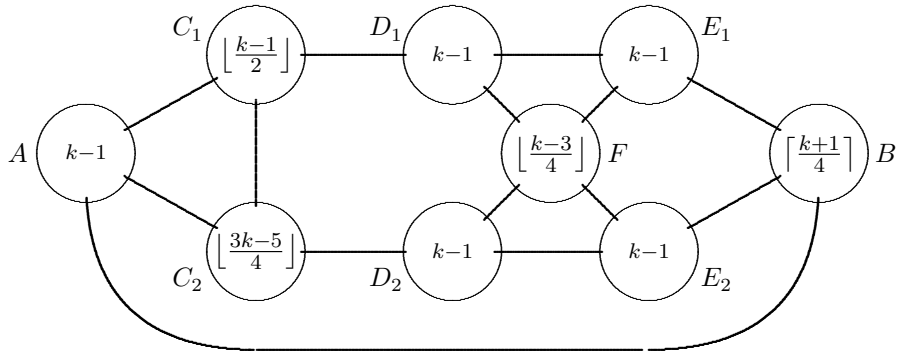

Figure 3. A graph G with $\alpha(G) = 3$, and no k-connected subgraph on at least $\lceil n/3 \rceil$ vertices.

We see that $\alpha(G) = 3$, even for $3 \leq k \leq 6$, when we have $F = \emptyset$. Also,

$$|G| = n = \begin{cases} \left\lceil \dfrac{27(k-1)}{4} \right\rceil - 1 & \text{if } k \text{ is odd,} \\ \left\lceil \dfrac{27(k-1)}{4} \right\rceil - 2 & \text{if } k \text{ is even.} \end{cases}$$

Now, $B \cup C_1 \cup F$ is a cut-set for G with at most $k-1$ vertices, so that any k-connected subgraph must be within either $H_1 = G[B \cup C_1 \cup D_1 \cup E_1 \cup F]$ or $H_2 = G - \{D_1 \cup E_1\}$. Since D_1 and E_1 are cut-sets for H_1, clearly, the largest k-connected subgraph of H_1 is $G[D_1 \cup E_1 \cup F]$. For H_2, we have $B \cup C_2$ is a cut-set with $k-1$ vertices. This means that any k-connected subgraph of H_2 must be within either $H_3 = G[B \cup C_2 \cup D_2 \cup E_2 \cup F]$ or $H_4 = G[A \cup B \cup C_1 \cup C_2]$. Since D_2 and E_2 are cut-sets for H_3, and A is a cut-set of H_4, all with $k-1$ vertices, it is clear that the largest k-connected subgraphs of H_3 and H_4 are $G[D_2 \cup E_2 \cup F]$ and $G[A \cup C_1 \cup C_2]$, respectively. Finally, it is easy to check that $G[A \cup C_1 \cup C_2]$ and $G[D_i \cup E_i \cup F]$, for $i = 1, 2$, all have at most $\lceil n/3 \rceil - 1$ vertices.

Proof of Theorem 6. In outline, the proof will be carried out as follows. We will proceed by contradiction, and assume that there is a graph G of order n satisfying the conditions of the theorem, but with the conclusion being false. Then, whenever we encounter a k-connected subgraph of G, we assume that it has fewer than $\lceil n/3 \rceil$ vertices. Also, whenever we encounter a subset with at least $\lceil n/3 \rceil$ vertices, then we assume that the subset does not induce a k-connected subgraph of G. We repeatedly use the fact that no four vertices of G can be independent. We end up arriving at a contradiction, where in certain cases, we obtain a k-connected subgraph of order at least $\lceil n/3 \rceil$.

We now present the proof of the theorem. Suppose that $k \geq 3$, and G is a graph with order n and independence number 3, where n is as in the statement of the theorem. Assume that the conclusion of the theorem is false, so that any k-connected subgraph of G has at most $\frac{n-1}{3}$ vertices. For the sake of clarity, we will only fully present the proof of the case with the lower bound of $n \geq \frac{27(k-1)}{4}$, for all $k \geq 3$, instead of the one as stated. This means that the remaining cases are:

(A) $k \equiv 0 \pmod 4$ and $n = \lceil \frac{27(k-1)}{4} \rceil - 1 = \frac{27(k-1)-1}{4}$,

(B) $k \equiv 2 \pmod 4$ and $n = \lceil \frac{27(k-1)}{4} \rceil - 1 = \frac{27(k-1)-3}{4}$.

To deal with cases (A) and (B), the crucial fact to note is that:

$$\left\lceil \frac{n}{3} \right\rceil - 1 = \begin{cases} \dfrac{n-2}{3} & \text{for case (A),} \\ \dfrac{n-3}{3} & \text{for case (B).} \end{cases}$$

By using this improved upper bound for the order of any k-connected subgraph of G, instead of the upper bound of $\frac{n-1}{3}$, the proofs for cases (A) and (B) will remain the same as the proof that we will present, only with minor adjustments to several inequalities. We will indicate in a footnote whenever an inequality requires an adjustment.

Thus, let $n \geq \frac{27(k-1)}{4}$. Since G itself is not k-connected, we can write

$$V(G) = A_1 \dot\cup A_2 \dot\cup S \text{ with } |S| \leq k-1, \quad |A_1|, |A_2| > 0, \quad E(A_1, A_2) = \emptyset,$$

and we may suppose, without loss of generality, that $G[A_2]$ is complete and $\alpha(G[A_1]) \leq 2$. Since $G[A_2]$ is k-connected if $|A_2| \geq k+1$, we must have $|A_2| \leq \frac{n-1}{3}$. Consequently, $|A_1| \geq \frac{2n+1}{3} - (k-1) > \frac{n}{3}$, so that $G[A_1]$ cannot be k-connected. Therefore, we can write

$$A_1 = A_{11} \dot\cup A_{12} \dot\cup S^* \text{ with } |S^*| \leq k-1, \quad 0 < |A_{11}| \leq |A_{12}|, \quad E(A_{11}, A_{12}) = \emptyset,$$

and where $G[A_{11}]$ and $G[A_{12}]$ are both complete. Now, $\alpha(G[A_1]) \leq 2$ implies that for all $v \in S^*$, we must have either $A_{11} \subset \Gamma(v)$ or $A_{12} \subset \Gamma(v)$. Hence, we can write

$$S^* = S_1^* \dot\cup S_2^*,$$

where

$$S_2^* = \{v \in S^* \mid A_{12} \subset \Gamma(v)\} \quad \text{and} \quad S_1^* = S^* \setminus S_2^*,$$

so that for all $v \in S_1^*$, we have $A_{11} \subset \Gamma(v)$.

Now, suppose that $|A_2| \leq k-1$. If $|A_{12}| \geq |A_{11}| \geq k$, then note that

$$\max\{|A_{11} \cup S_1^*|, |A_{12} \cup S_2^*|\} \geq \frac{n-2(k-1)}{2} = \frac{n}{2} - (k-1) > \frac{n}{3},$$

and so either $G[A_{11} \cup S_1^*]$ or $G[A_{12} \cup S_2^*]$ is a k-connected subgraph with more than $\frac{n}{3}$ vertices, a contradiction. Otherwise, we have $|A_{11}| \leq k-1$, so that $|A_{12}| \geq n - 4(k-1) > \frac{n}{3}$, and $G[A_{12}]$ is a desired k-connected subgraph, another contradiction. Therefore, we have $|A_2| \geq k$. We define $S_3 \subset S$ such that $G[A_2 \cup S_3]$ is k-connected and $|S_3|$ is maximum. In the case that $|A_2| = k$ and no $S_3 \subset S$ makes $G[A_2 \cup S_3]$ k-connected, we let $S_3 = \emptyset$. Clearly, we have[1]

$$|A_2 \cup S_3| \leq \frac{n-1}{3}. \tag{4}$$

Thus, we record the following useful fact, by utilising the maximality of $|S_3|$:

$$|A_2| \geq k, \text{ and for all } v \in S \setminus S_3,\ e(v, A_2) \leq k-1, \text{ so there exists } w \in A_2 \setminus \Gamma(v). \tag{5}$$

[1] $|A_2 \cup S_3| \leq \frac{n-2}{3}$ if (A) holds; $|A_2 \cup S_3| \leq \frac{n-3}{3}$ if (B) holds.

Since $\alpha(G) = 3$, this means that for all $v \in S \setminus S_3$, we must have either $A_{11} \subset \Gamma(v)$ or $A_{12} \subset \Gamma(v)$. We can write

$$S = S_1 \dot\cup S_2 \dot\cup S_3,$$

where

$$S_2 = \{v \in S \setminus S_3 \mid A_{12} \subset \Gamma(v)\} \quad \text{and} \quad S_1 = S \setminus (S_2 \cup S_3),$$

so that for all $v \in S_1$, we have $A_{11} \subset \Gamma(v)$. We note that:

$$\text{Every vertex of } A_{11} \text{ (resp. } A_{12}) \text{ dominates } A_{11} \cup S_1 \cup S_1^* \text{ (resp. } A_{12} \cup S_2 \cup S_2^*). \tag{6}$$

The situation at this point is illustrated in Figure 4.

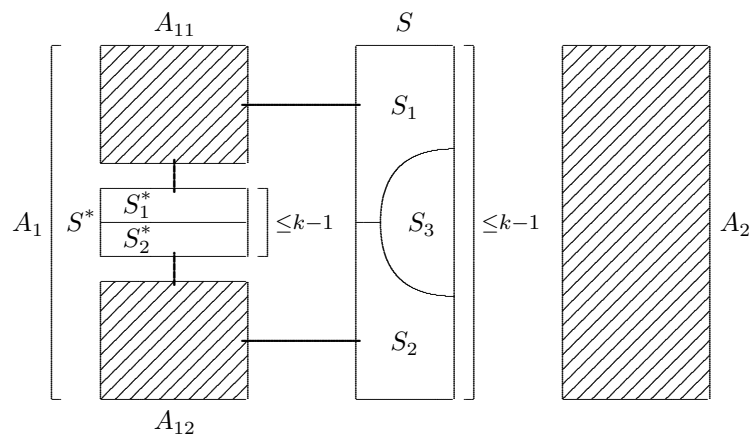

Figure 4. Illustration for the proof of Theorem 6. Shaded sets indicate complete graphs.

Now, in addition to (5), it is convenient to record bounds on the sizes of some other subsets that we have defined. Suppose that $|A_{12}| \geq |A_{11}| \geq k$. In view of (4),

$$\max\{|A_{11} \cup S_1 \cup S_1^*|, |A_{12} \cup S_2 \cup S_2^*|\} \geq \frac{n - |A_2 \cup S_3|}{2} \geq \frac{2n+1}{6} > \frac{n}{3}.$$

By (6), this means that either $G[A_{11} \cup S_1 \cup S_1^*]$ or $G[A_{12} \cup S_2 \cup S_2^*]$ is a desired k-connected subgraph, a contradiction. Therefore, we have

$$|A_{11}| \leq k - 1. \tag{7}$$

From (4) and (7), we have[2]

$$|A_{12}| \geq n - |A_2 \cup S_3| - |A_{11}| - |S| - |S^*| \geq \frac{2n+1}{3} - 3(k-1) > \frac{3(k-1)}{2} \geq k. \tag{8}$$

In view of (6) and (8), we see that $G[A_{12} \cup S_2 \cup S_2^*]$ is k-connected. We define N to be a subset of $S_1 \cup S_1^*$ such that

$$H = G[N \cup A_{12} \cup S_2 \cup S_2^*] \tag{9}$$

[2] $|A_{12}| \geq \frac{2n+2}{3} - 3(k-1) > \frac{3(k-1)}{2} \geq k$ if (A) or (B) holds.

is k-connected and $|N|$ is maximum. Thus[3],

$$|H| \leq \frac{n-1}{3}. \tag{10}$$

Let

$$L = S_1 \setminus N \quad \text{and} \quad L^* = S_1^* \setminus N,$$

and note that

$$e(v, N \cup A_{12} \cup S_2 \cup S_2^*) \leq k-1 \text{ for all } v \in L \cup L^*. \tag{11}$$

By (4) and (10), we have[4]

$$|A_{11} \cup L \cup L^*| \geq \frac{n+2}{3} > \frac{9(k-1)}{4}. \tag{12}$$

At this point, we briefly remark that, by (7) and (12), we have $\frac{n+2}{3} \leq |A_{11} \cup L \cup L^*| \leq 3(k-1)$, which gives $n < 9(k-1)$. Thus, we have proved the version of Theorem 6 with the much weaker lower bound of $n \geq 9(k-1)$, since we would now have a contradiction. Indeed, the argument up to this point is very similar to the proofs of Theorems 2 and 4.

By (12), we may assume that $G[A_{11} \cup L \cup L^*]$ is not k-connected. We may write

$$A_{11} \cup L \cup L^* = B_1 \dot\cup B_2 \dot\cup T' \text{ with } |T'| \leq k-1, \quad |B_1|, |B_2| > 0, \quad E(B_1, B_2) = \emptyset.$$

Let $B = B_1 \cup B_2$. From (6), we see that every vertex in A_{11} dominates $A_{11} \cup L \cup L^*$. This means that we must have

$$A_{11} \subset T' \quad \text{and} \quad B \subset L \cup L^*. \tag{13}$$

Using $|T'| \leq k-1$, $|L| \leq k-1$, (12) and (13), we have[5]

$$|B| > \frac{5(k-1)}{4}, \quad \text{and} \quad |B \cap L^*| > \frac{k-1}{4}, \tag{14}$$

so that $B \cap L^* \neq \emptyset$. Furthermore, since $G[A_1 \cup S_1 \cup S_2]$ is not k-connected, we can write

$$A_1 \cup S_1 \cup S_2 = A \dot\cup A' \dot\cup T \text{ with } |T| \leq k-1, \quad |A|, |A'| > 0, \quad E(A, A') = \emptyset.$$

Since $|B| \geq k$ by (14), and $|T| \leq k-1$, we have $B \cap (A \cup A') \neq \emptyset$. By symmetry of (A, A') and (B_1, B_2), we may assume that $A \cap B_1 \neq \emptyset$. From (13), we have

$$\emptyset \neq A \cap B_1 \subset L \cup L^*. \tag{15}$$

The following three lemmas will be very useful in our subsequent arguments.

Lemma 8. *There cannot exist three independent vertices x, y, z with $x, y \in A_1$ and $z \in A_1 \cup S_1 \cup S_2$.*

Proof. If $x, y, z \in A_1$, then $\alpha(G[A_1]) \leq 2$ implies that $\{x, y, z\}$ cannot be independent. Otherwise, $x, y \in A_1$ and $z \in S_1 \cup S_2$. By (5), we can find $w \in A_2 \setminus \Gamma(z)$. If $\{x, y, z\}$ is an independent set, then so is $\{x, y, z, w\}$, and this contradicts $\alpha(G) = 3$. □

[3] $|H| \leq \frac{n-2}{3}$ if (A) holds; $|H| \leq \frac{n-3}{3}$ if (B) holds.

[4] $|A_{11} \cup L \cup L^*| \geq \frac{n+4}{3} = \frac{9k-4}{4}$ if (A) holds; $|A_{11} \cup L \cup L^*| \geq \frac{n+6}{3} = \frac{9k-2}{4}$ if (B) holds.

[5] $|B| \geq \frac{5k}{4}$ and $|B \cap L^*| \geq \frac{k+4}{4}$ if (A) holds; $|B| \geq \frac{5k+2}{4}$ and $|B \cap L^*| \geq \frac{k+6}{4}$ if (B) holds.

Lemma 9. *Suppose that there exist $x \in L^*$ and $z \in L \cup L^*$ with $xz \notin E(G)$, and at least one of x and z is in A. Then $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$.*

Proof. Assume that $|A' \cap (A_{12} \cup S_2^*)| \geq k$. If $x \in A$, then in view of (11), we can find $y \in (A' \cap (A_{12} \cup S_2^*)) \setminus \Gamma(z)$. But then $\{x, y, z\}$ is independent, contradicting Lemma 8. A similar argument holds if $z \in A$. □

Lemma 10. *Suppose that $B_2 \cap L^* = \emptyset$. Then $|A_{12} \cup S_2^*| \leq 2(k-1)$.*

Proof. Note that since $B \cap L^* \neq \emptyset$ by (14), and $B_2 \cap L^* = \emptyset$, we can pick $x \in B_1 \cap L^*$. Also, pick $z \in B_2 \subset L \cup L^*$ by (13). Then, we see that $A_{12} \cup S_2^* \subset \Gamma(x) \cup \Gamma(z)$, or else we can find $y \in (A_{12} \cup S_2^*) \setminus (\Gamma(x) \cup \Gamma(z))$ with $\{x, y, z\}$ independent, contradicting Lemma 8. By (11), we have $|A_{12} \cup S_2^*| \leq 2(k-1)$, as required. □

We now derive some key properties for the sets A, A' and T.

Claim 11. *$A' \cap A_{11} = \emptyset$.*

Proof. Suppose that $A' \cap A_{11} \neq \emptyset$. In view of (6), since every vertex of A_{11} dominates $A_{11} \cup S_1 \cup S_1^*$, we have $A_{11} \cup S_1 \cup S_1^* \subset A' \cup T$. Thus $B_1 \subset A_{11} \cup L \cup L^* \subset A_{11} \cup S_1 \cup S_1^* \subset A' \cup T$, and this contradicts $A \cap B_1 \neq \emptyset$ in (15). □

Claim 12. *$A \cap A_{12} = \emptyset$.*

Proof. Suppose that $A \cap A_{12} \neq \emptyset$. It then follows from (6) that $A_{12} \cup S_2 \cup S_2^* \subset A \cup T$. If $A \cap A_{11} \neq \emptyset$, then again by (6), we have $A_1 \cup S_1 \cup S_2 \subset A \cup T$, a contradiction. Thus, $A \cap A_{11} = \emptyset$, and with Claim 11, this implies that $A_{11} \subset T$. Together with $|T| \leq k-1$, and $|A_{11} \cup L^*| \geq k$ by (12), this means that $(A \cup A') \cap L^* \neq \emptyset$. Assume now that $|A \cap A_{12}| \geq k$. If $x \in A \cap L^*$, then we pick $y \in (A \cap A_{12}) \setminus \Gamma(x)$ by (11), and $z \in A'$; and if $x \in A' \cap L^*$, then we pick $z \in A \cap B_1 \subset L \cup L^*$ by (15), and $y \in (A \cap A_{12}) \setminus \Gamma(z)$ by (11). In either case, $\{x, y, z\}$ is independent, contradicting Lemma 8. Consequently, $|A \cap A_{12}| \leq k-1$.

Now, since $A \cap A_{12} \neq \emptyset$ and $G[A_{12}]$ is complete, we have $A' \cap A_{12} = \emptyset$. Since $A_{11} \subset T$, we have $|A_{11} \cup A_{12}| \leq |T| + |A \cap A_{12}| \leq 2(k-1)$. Thus $|A_2| \geq n - 4(k-1) > \frac{n}{3}$, contradicting (4). □

Claim 13. *$A' \cap A_{12} \neq \emptyset$, and $A \subset A_{11} \cup S_1 \cup S_1^*$.*

Proof. Since $A \cap A_{12} = \emptyset$ by Claim 12, and $|T| \leq k-1$, we have $|A' \cap A_{12}| > \frac{k-1}{2}$ by (8), and the first part follows. The second part then follows since $A' \cap A_{12} \neq \emptyset$ and (6) imply $A \cap (A_{12} \cup S_2 \cup S_2^*) = \emptyset$. □

Claim 14. *$A \cap A_{11} \neq \emptyset$, and $A' \subset A_{12} \cup S_2 \cup S_2^*$.*

Proof. It suffices to prove the first part, since then $A \cap A_{11} \neq \emptyset$ and (6) imply $A' \cap (A_{11} \cup S_1 \cup S_1^*) = \emptyset$. Suppose that $A \cap A_{11} = \emptyset$, so that Claim 11 implies $A_{11} \subset T$. We first prove that

$$|A' \cap A_{12}| \geq k. \tag{16}$$

Otherwise, suppose that $|A' \cap A_{12}| \leq k-1$. Since $A_{11} \subset T$, and $A \cap A_{12} = \emptyset$ by Claim 12, we have $|A_{11} \cup A_{12}| \leq |T| + |A' \cap A_{12}| \leq 2(k-1)$. Thus $|A_2| \geq n - 4(k-1) > \frac{n}{3}$, contradicting (4). Thus (16) holds.

Now, we consider two cases.

Case 1. $B_2 \cap L^* \neq \emptyset$.

Taking $x \in B_2 \cap L^*$, and $z \in A \cap B_1 \subset L \cup L^*$ by (15), we see from Lemma 9 that $|A' \cap A_{12}| \leq k-1$. This contradicts (16).

Case 2. $B_2 \cap L^* = \emptyset$.

We first show that $|A_2| \leq 2(k-1)$. Pick $u \in A \cap B_1 \subset L \cup L^*$ by (15), $z \in B_2 \subset L \cup L^*$ by (13), and $v \in (A' \cap A_{12}) \setminus \Gamma(z)$ by (11) and (16). Then $\{v, u, z\}$ is independent, and so $u, z \in L$, otherwise we have a contradiction to Lemma 8. We have $A_2 \subset \Gamma(u) \cup \Gamma(z)$, or else we can pick $w \in A_2 \setminus (\Gamma(u) \cup \Gamma(z))$ with $\{v, u, z, w\}$ independent, contradicting $\alpha(G) = 3$. By (5), we have $|A_2| \leq 2(k-1)$, as required.

Next, we show that $B_1 \cap L^* \subset T$. Otherwise, if we have $x \in A \cap B_1 \cap L^*$, then we pick $z \in B_2 \subset L \cup L^*$ by (13); and if we have $x \in A' \cap B_1 \cap L^*$, then we pick $z \in A \cap B_1 \subset L \cup L^*$ by (15). In either case, Lemma 9 implies $|A' \cap A_{12}| \leq k-1$, and this contradicts (16). Thus $(A \cup A') \cap B_1 \cap L^* = \emptyset$, and $B_1 \cap L^* \subset T$ as required.

Together with $B_2 \cap L^* = \emptyset$, we have $|T \cap L^*| \geq |B_1 \cap L^*| = |B \cap L^*|$. In view of $A_{11} \subset T$, $|T| \leq k-1$, and (14), we have[6]

$$|A_{11}| \leq |T| - |T \cap L^*| \leq |T| - |B \cap L^*| < \frac{3(k-1)}{4}.$$

Since $B_2 \cap L^* = \emptyset$, by Lemma 10, we have $|A_{12}| \leq |A_{12} \cup S_2^*| \leq 2(k-1)$. Thus[7]

$$n = |A_{11}| + |A_{12}| + |A_2| + |S| + |S^*| < \frac{27(k-1)}{4}.$$

This contradicts the hypothesis on n. □

Claim 15. $N \subset T$.

Proof. We have $A' \cap N = \emptyset$ since $A' \subset A_{12} \cup S_2 \cup S_2^*$ from Claim 14. Also, if $v \in A \cap N \neq \emptyset$, then since $A' \cap A_{12} \neq \emptyset$ by Claim 13, we have $T \cap V(H)$ is a cut-set of H with fewer than k vertices which separates v from a vertex of $A' \cap A_{12}$, contradicting that H is k-connected. Therefore, $A \cap N = \emptyset$, and $N \subset T$. □

To summarise, at this point we know the following, which one should constantly keep in mind for the rest of the proof of Theorem 6.

$$A \subset A_{11} \cup L \cup L^*, \quad A' \subset A_{12} \cup S_2 \cup S_2^*, \quad A' \cap A_{12} \neq \emptyset, \quad N \subset T. \tag{17}$$

The first fact in (17) follows from $A \subset A_{11} \cup S_1 \cup S_1^*$ and $N \subset T$. We also note that:

$$A \text{ (resp. } A') \text{ is disjoint from any subset of } A_{12} \cup S_2 \cup S_2^* \text{ (resp. } A_{11} \cup S_1 \cup S_1^*). \tag{18}$$

[6] $|A_{11}| \leq \frac{3k-8}{4}$ if (A) or (B) holds.
[7] $n \leq \frac{27(k-1)-5}{4}$ if (A) or (B) holds.

We now split the proof of Theorem 6 into two cases. Recall that $B \cap L^* \neq \emptyset$ from (14) and $A \cap B_1 \neq \emptyset$ from (15).

Case 1. $B_2 \cap L^* \neq \emptyset$.

Subcase 1.1. $A \cap B_1 \cap L^* \neq \emptyset$.

Suppose that $|A'| \geq k$. We can pick $x \in A \cap B_1 \cap L^*$ and $y \in B_2 \cap L^*$, and since $A' \subset A_{12} \cup S_2 \cup S_2^*$ by (17), we can pick $z \in A' \setminus \Gamma(y)$ by (11). But then $\{x, y, z\}$ is independent, contradicting Lemma 8. Thus $|A'| \leq k-1$, and hence by (4),

$$|A| \geq \frac{2n+1}{3} - |A'| - |T| \geq \frac{2n+1}{3} - 2(k-1) > \frac{n}{3}.$$

Now, we have $A \subset A_{11} \cup L \cup L^*$ by (17). Since $|L| \leq k-1$, we have $|A \setminus L| > \frac{n}{3} - (k-1) > k-1$, so that $|A \setminus L| \geq k$. Also, $G[A \setminus L]$ is a clique. Otherwise we can pick $x, y \in A \setminus L \subset A_{11} \cup L^*$ with $xy \notin E(G)$, and $z \in A'$, so that $\{x, y, z\}$ is independent, contradicting Lemma 8. Finally, every vertex of $A \cap L$ dominates $A \setminus L$. Otherwise, we can pick $x \in A \setminus L$ and $z \in A \cap L$ with $xz \notin E(G)$, and $y \in A' \cap A_{12}$ by (17), so that $\{x, y, z\}$ is independent, contradicting Lemma 8. Therefore, $G[A]$ is a k-connected subgraph with more than $\frac{n}{3}$ vertices, a contradiction.

Subcase 1.2. $A \cap B_1 \cap L^* = \emptyset$.

Pick $z \in A \cap B_1 \subset L$ by (15). If we have $x \in A \cap B_2 \cap L^*$, then we can pick $y \in A' \cap A_{12}$ by (17), so that $\{x, y, z\}$ is independent, contradicting Lemma 8. Consequently, we have $A \cap B_1 \cap L^* = A \cap B_2 \cap L^* = \emptyset$, so $A \cap B \cap L^* = \emptyset$. Since $A' \cap B \cap L^* = \emptyset$ by (18), we have $B \cap L^* \subset T$. Next, we pick $x \in B_2 \cap L^*$, and with z, we obtain $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$ by Lemma 9. Therefore, since $A \cap (A_{12} \cup S_2^*) = \emptyset$ by (18), $N \subset T$ by (17), and $B \cap L^* \subset T$, we have

$$|A_{12} \cup S_2^*| + |N| + |B \cap L^*| \leq |A' \cap (A_{12} \cup S_2^*)| + |T| \leq 2(k-1). \tag{19}$$

Now, recall that $A_{11} \subset T'$ by (13). Then we see that $A_1 \cup S_1 \cup S_2$ can be written as follows.

$$A_1 \cup S_1 \cup S_2 = (A_{12} \cup S_2^*) \,\dot\cup\, N \,\dot\cup\, (B \cap L^*) \,\dot\cup\, T' \,\dot\cup\, (B \cap L) \,\dot\cup\, S_2.$$

Hence by (4) and (19),

$$|T'| + |B \cap L| + |S_2| \geq \frac{2n+1}{3} - 2(k-1) > 2(k-1).$$

On the other hand, $T' \,\dot\cup\, (B \cap L) \,\dot\cup\, S_2 \subset T' \cup S$, so that $|T'| + |B \cap L| + |S_2| \leq |T'| + |S| \leq 2(k-1)$. We have a contradiction. This completes the proof of Subcase 1.2 and also of Case 1.

Case 2. $B_2 \cap L^* = \emptyset$.

Note that by (13) and (14), we have

$$B_2 \subset L, \quad \text{and} \quad B_1 \cap L^* = B \cap L^* \text{ is non-empty.} \tag{20}$$

We begin by deriving bounds for the sizes of certain sets.

Claim 16. $|A_2| \leq 2(k-1)$.

Proof. Suppose that $|A_2| > 2(k-1)$. We first note that:

$$\text{There cannot exist } x \in A_1 \text{ and } u, z \in A_1 \cup S_1 \cup S_2 \text{ with } \{x, u, z\} \text{ independent.} \tag{21}$$

Otherwise, in view of Lemma 8, we have $u, z \in S_1 \cup S_2$, and by (5), we can find $w \in A_2 \setminus (\Gamma(u) \cup \Gamma(z))$. But then $\{x, u, z, w\}$ is independent, contradicting $\alpha(G) = 3$.

This means that $G[A]$ is a clique, since if we have $u, z \in A$ and $uz \notin E(G)$, then taking $x \in A' \cap A_{12}$ by (17), we have $\{x, u, z\}$ is independent, contradicting (21).

Next, we prove that $B_1 \cap L^* \subset T$. Note first that $A' \cap B_1 \cap L^* = \emptyset$ by (18). Suppose that we have $A \cap B_1 \cap L^* \neq \emptyset$. If $|A'| \geq k$, then we can pick $x \in A \cap B_1 \cap L^*$, $u \in B_2 \subset L$ by (20), and $z \in A' \setminus \Gamma(u)$ by (11) and $A' \subset A_{12} \cup S_2 \cup S_2^*$ in (17), so that $\{x, u, z\}$ is independent, contradicting (21). Thus, we have $|A'| \leq k-1$. Since $|A_1 \cup S_1 \cup S_2| \geq \frac{2n+1}{3}$ by (4), and $|T| \leq k-1$, we have $G[A]$ is a k-connected subgraph on at least $\frac{2n+1}{3} - 2(k-1) > \frac{n}{3}$ vertices, a contradiction. Therefore, $(A \cup A') \cap B_1 \cap L^* = \emptyset$, and $B_1 \cap L^* \subset T$ follows.

Now, we have $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$. Otherwise, we can take $u \in B_2 \subset L$ by (20), $x \in (A' \cap (A_{12} \cup S_2^*)) \setminus \Gamma(u)$ by (11), and $z \in A \cap B_1$ by (15), so that $\{x, u, z\}$ is independent, contradicting (21).

From (18), we have $A \cap (A_{12} \cup S_2^*) = \emptyset$. Thus by (8),

$$|T \cap (A_{12} \cup S_2^*)| = |A_{12} \cup S_2^*| - |A' \cap (A_{12} \cup S_2^*)| > \frac{k-1}{2}.$$

Together with $B_1 \cap L^* \subset T$ and $|T| \leq k-1$, we have $|B_1 \cap L^*| < \frac{k-1}{2}$. If $k = 3$, then we have a contradiction to $B_1 \cap L^* \neq \emptyset$ in (20), so let $k \geq 4$ for the remainder of the proof. Since $A_{11} \subset T'$ by (13), together with (20), we have $A_{11} \cup L \cup L^* = (B \cap L) \dot\cup (B \cap L^*) \dot\cup T' = (B \cap L) \dot\cup (B_1 \cap L^*) \dot\cup T'$. Therefore by (12),

$$|B \cap L| = |A_{11} \cup L \cup L^*| - |T'| - |B_1 \cap L^*| > \frac{3(k-1)}{4},$$

which implies $|S_2| \leq |S| - |B \cap L| < \frac{k-1}{4}$. Since $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$, and $A' \subset A_{12} \cup S_2 \cup S_2^*$ by (17), we have $|A'| < \frac{5(k-1)}{4}$. Thus by (4), $G[A]$ is a k-connected subgraph on at least $\frac{2n+1}{3} - |A'| - |T| > \frac{2n+1}{3} - \frac{9(k-1)}{4} > \frac{n}{3}$ vertices, a contradiction. □

By (7) and Claim 16, we have[8]

$$|A_{12}| = n - |A_{11}| - |S^*| - |S| - |A_2| \geq \frac{7(k-1)}{4}. \tag{22}$$

Also, since $B_2 \cap L^* = \emptyset$, we have $|A_{12} \cup S_2^*| \leq 2(k-1)$ by Lemma 10, and thus[9]

$$|A_{11}| = n - |A_2| - |S| - |S_1^*| - |A_{12} \cup S_2^*| \geq \frac{3(k-1)}{4}. \tag{23}$$

Now, we proceed by proving several claims.

Claim 17. $A \cap B_2 = \emptyset$.

[8] $|A_{12}| \geq \frac{7k-8}{4}$ if (A) holds; $|A_{12}| \geq \frac{7k-10}{4}$ if (B) holds.
[9] $|A_{11}| \geq \frac{3k-4}{4}$ if (A) holds; $|A_{11}| \geq \frac{3k-6}{4}$ if (B) holds.

Proof. Suppose that we have $z \in A \cap B_2 \subset L$ by (20). Note that $A' \cap B_1 \cap L^* = \emptyset$ by (18). If we have $x \in A \cap B_1 \cap L^*$, then pick $y \in A' \cap A_{12}$ by (17), so that $\{x, y, z\}$ is independent, contradicting Lemma 8. Hence, we have $(A \cup A') \cap B_1 \cap L^* = \emptyset$, and $B \cap L^* = B_1 \cap L^* \subset T$ by (20). Next, taking $x \in B_1 \cap L^*$ by (20), and z, and applying Lemma 9, we have $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$. Finally, $A \cap (A_{12} \cup S_2^*) = \emptyset$ by (18). Together with (14) and $|T| \leq k-1$, we have[10]

$$\begin{aligned}|A_{12}| &\leq |A_{12} \cup S_2^*| = |A' \cap (A_{12} \cup S_2^*)| + |T \cap (A_{12} \cup S_2^*)| \\ &\leq |A' \cap (A_{12} \cup S_2^*)| + |T| - |T \cap L^*| \\ &\leq |A' \cap (A_{12} \cup S_2^*)| + |T| - |B \cap L^*| < \frac{7(k-1)}{4},\end{aligned}$$

which contradicts (22). □

Since $A' \cap B = \emptyset$ by (18), together with Claim 17, we have $B = (A \cap B) \dot\cup (T \cap B) = (A \cap B_1) \dot\cup (T \cap B)$, and thus

$$|B| = |A \cap B_1| + |T \cap B|. \tag{24}$$

Claim 18. $B_1 \cap L^* \subset T$.

Proof. Since $A' \cap B_1 \cap L = \emptyset$ by (18), it suffices to prove that $A \cap B_1 \cap L^* = \emptyset$. Assume that $A \cap B_1 \cap L^* \neq \emptyset$. Fix $x \in A \cap B_1 \cap L^*$, and $z \in B_2 \subset L$ by (20). By Lemma 9, we have

$$|A' \cap A_{12}| \leq |A' \cap (A_{12} \cup S_2^*)| \leq k-1. \tag{25}$$

Also, $A \cap (A_{12} \cup S_2^*) = \emptyset$ by (18). Together with (22), (25), and $|T| \leq k-1$, we have[11]

$$\begin{aligned}|T \setminus (A_{12} \cup S_2^*)| &= |T| - |T \cap (A_{12} \cup S_2^*)| \\ &= |T| - (|A_{12} \cup S_2^*)| - |A' \cap (A_{12} \cup S_2^*)|) \\ &\leq |T| + |A' \cap (A_{12} \cup S_2^*)| - |A_{12}| \leq \frac{k-1}{4}.\end{aligned} \tag{26}$$

Next, we show that $G[A \cup A_{11}]$ is k-connected. Assume otherwise. Note that every vertex of $A \cap L^*$ dominates $A \cup A_{11}$. Otherwise, we can pick $x' \in A \cap L^*$ and $z' \in A \cup A_{11}$ with $x'z' \notin E(G)$, and $y \in A' \cap A_{12}$ by (17), so that $\{x', y, z'\}$ is independent, contradicting Lemma 8. Since $A \cup A_{11} \subset A_{11} \cup L \cup L^*$ by (17), together with (6), $(A \cap L^*) \cup A_{11}$ consists of dominating vertices of $A \cup A_{11}$ and so must be contained in a cut-set for $G[A \cup A_{11}]$. Thus, we have $|A \cap L^*| + |A_{11}| \leq k-1$. Also, $A' \cap L^* = \emptyset$ by (18), so that $|L^*| = |A \cap L^*| + |T \cap L^*|$. Again by (18), we have $A \cap A_{12} = \emptyset$, so that $|A_{12}| = |A' \cap A_{12}| + |T \cap A_{12}|$. Together with (12), (25), and $|T| \leq k-1$, we have[12]

$$\begin{aligned}|A_{12}| &= |A' \cap A_{12}| + |T \cap A_{12}| \leq |A' \cap A_{12}| + |T| - |T \cap L^*| \\ &= |A' \cap A_{12}| + |T| - (|L^*| - |A \cap L^*|) \\ &= |A' \cap A_{12}| + |T| + (|A \cap L^*| + |A_{11}|) - |A_{11} \cup L^*| < \frac{7(k-1)}{4},\end{aligned}$$

[10] $|A_{12}| \leq \frac{7k-12}{4}$ if (A) or (B) holds.
[11] $|T \setminus (A_{12} \cup S_2^*)| \leq \frac{k}{4}$ if (A) holds; $|T \setminus (A_{12} \cup S_2^*)| \leq \frac{k+2}{4}$ if (B) holds.
[12] $|A_{12}| \leq \frac{7k-12}{4}$ if (A) or (B) holds.

which contradicts (22). Therefore, we have $G[A \cup A_{11}]$ is k-connected. In particular[13]

$$|A| \leq |A \cup A_{11}| \leq \frac{n-1}{3}. \tag{27}$$

Now, we have $A' \cup A_{12} \cup S_2^* \subset A_{12} \cup S_2 \cup S_2^*$ by (17). Suppose that $A' \cup A_{12} \cup S_2^* \subset \Gamma(x) \cup \Gamma(z)$. Then $|A' \cup A_{12} \cup S_2^*| \leq 2(k-1)$ by (11). Also, note that

$$A \dot{\cup} A' \dot{\cup} T = A \cup (T \setminus (A_{12} \cup S_2^*)) \cup (A' \cup A_{12} \cup S_2^*).$$

Therefore, together with (4), (26) and (27), we have[14]

$$\frac{2n+1}{3} \leq |A| + |T \setminus (A_{12} \cup S_2^*)| + |A' \cup A_{12} \cup S_2^*| \leq \frac{n-1}{3} + \frac{9(k-1)}{4},$$

which implies[15] $n < \frac{27(k-1)}{4}$, contradicting our hypothesis.

This means that we can find $u \in (A' \cup A_{12} \cup S_2^*) \setminus (\Gamma(x) \cup \Gamma(z))$, and thus $\{x, u, z\}$ is independent. Since $A' \cup A_{12} \cup S_2^* \subset A_{12} \cup S_2 \cup S_2^*$, we have $u \in (A' \cap S_2) \setminus (\Gamma(x) \cup \Gamma(z))$, otherwise we have a contradiction to Lemma 8. Now define

$$S_3' = S_3 \setminus (\Gamma(z) \cup \Gamma(u)).$$

In view of $z, u \in S \setminus S_3$ and the maximality of S_3, we have $e(z, A_2 \cup (S_3 \setminus S_3')) \leq k-1$ and $e(u, A_2 \cup (S_3 \setminus S_3')) \leq k-1$. Moreover, we must have $A_2 \cup (S_3 \setminus S_3') \subset \Gamma(z) \cup \Gamma(u)$, otherwise we can take $w \in A_2 \setminus (\Gamma(z) \cup \Gamma(u))$, and $\{x, z, u, w\}$ is independent, contradicting $\alpha(G) = 3$. Consequently,

$$|A_2 \cup (S_3 \setminus S_3')| \leq 2(k-1). \tag{28}$$

Now, if $S_3' \neq \emptyset$ and $v \in S_3'$, then v dominates $A \cap B_1$. Otherwise, if there exists $v' \in A \cap B_1$ with $vv' \notin E(G)$, then $\{z, u, v, v'\}$ is independent, contradicting $\alpha(G) = 3$. Since $T \cap B \subset T \setminus (A_{12} \cup S_2^*)$, by (14), (24) and (26), we have

$$|A \cap B_1| = |B| - |T \cap B| \geq |B| - |T \setminus (A_{12} \cup S_2^*)| > k-1,$$

which implies $|A \cap B_1| \geq k$. Since $G[A \cup A_{11}]$ is k-connected and $A \cap B_1 \subset A \cup A_{11}$, we have $G[A \cup A_{11} \cup S_3']$ is also k-connected (whether S_3' is empty or non-empty).

Finally, from the definition of the subgraph H in (9), and $A' \cap (A_{11} \cup S_1 \cup S_1^*) = \emptyset$ by (18), we have $A_{11} \cup S_1 \cup S_1^* \subset A \cup (T \setminus (A_{12} \cup S_2^*))$, and $A_{12} \cup S_2 \cup S_2^* \subset V(H)$, and thus

$$V(G) = (A \cup A_{11} \cup S_3') \cup V(H) \cup (T \setminus (A_{12} \cup S_2^*)) \cup (A_2 \cup (S_3 \setminus S_3')).$$

By (10), (26) and (28), we have[16]

$$\begin{aligned} |A \cup A_{11} \cup S_3'| &\geq n - |H| - |T \setminus (A_{12} \cup S_2^*)| - |A_2 \cup (S_3 \setminus S_3')| \\ &\geq \frac{2n+1}{3} - \frac{9(k-1)}{4} \geq \frac{n}{3}. \end{aligned}$$

Hence, $G[A \cup A_{11} \cup S_3']$ is a k-connected subgraph on at least $\frac{n}{3}$ vertices, a contradiction. □

[13] $|A| \leq \frac{n-2}{3}$ if (A) or (B) holds.

[14] $\frac{2n+2}{3} \leq |A| + |T \setminus (A_{12} \cup S_2^*)| + |A' \cup A_{12} \cup S_2^*| \leq \frac{n-2}{3} + \frac{9k-6}{4}$ if (A) or (B) holds.

[15] $n \leq \frac{27(k-1)-7}{4}$ if (A) or (B) holds.

[16] $|A \cup A_{11} \cup S_3'| \geq \frac{2n+2}{3} - \frac{9k-8}{4} > \frac{n}{3}$ if (A) holds; $|A \cup A_{11} \cup S_3'| \geq \frac{2n+3}{3} - \frac{9k-6}{4} \geq \frac{n}{3}$ if (B) holds.

Now, by (14) and (24), we have[17]

$$|A_1 \cap B| = |B| - |T \cap B| > \frac{k-1}{4}. \tag{29}$$

In view of $B_2 \cap L^* = \emptyset$ and Claim 18, we have $L^* = (B_1 \cap L^*) \cup (T' \cap L^*) \subset T \cup T'$. Also, since $A \cap (A_{12} \cup S_2^*) = \emptyset$ by (18), we have $A_{12} \cup S_2^* \subset T \cup (A' \cap (A_{12} \cup S_2^*))$. Since $A_{11} \subset T'$ by (13), and $N \subset T$ by (17), we have

$$A_1 \subset A_{11} \cup L^* \cup N \cup A_{12} \cup S_2^* \subset T \cup T' \cup (A' \cap (A_{12} \cup S_2^*)).$$

If $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$, then together with Claim 16, we have

$$n = |A_1| + |A_2| + |S| \leq |T| + |T'| + |A' \cap (A_{12} \cup S_2^*)| + |A_2| + |S| \leq 6(k-1),$$

a contradiction. Consequently,

$$|A' \cap (A_{12} \cup S_2^*)| \geq k. \tag{30}$$

Now, recall that $B_2 \subset L$ from (20). For the rest of the proof of Theorem 6, we fix the vertex z, where

$$z \in B_2 \subset L. \tag{31}$$

Define

$$M = \Gamma(z) \cap N \cap S_1^*, \quad M' = (N \cap S_1^*) \setminus M, \quad Q = A \cup A_{11} \cup L^* \cup M'. \tag{32}$$

Claim 19. *$G[Q]$ is either a k-connected subgraph, or a clique on k vertices.*

Proof. We first note that:

$$\text{Every vertex of } L^* \text{ dominates } A_{11} \cup (A \cap B_1). \tag{33}$$

Indeed by (6), every vertex of L^* dominates A_{11}. Also, if we have $x \in L^*$, and $z' \in A \cap B_1 \subset L \cup L^*$ by (15), with $xz' \notin E(G)$, then Lemma 9 implies $|A' \cap (A_{12} \cup S_2^*)| \leq k-1$, which contradicts (30).

Next, we show that $G_1 = G[A_{11} \cup (B_1 \cap L^*) \cup (A \cap B_1)] \subset G[Q]$ is complete. Obviously by (6), every vertex of A_{11} dominates $V(G_1)$. In view of (33), it suffices to show that both $G[B_1 \cap L^*]$ and $G[A \cap B_1]$ are complete. If $x, y \in B_1 \cap L^*$ and $xy \notin E(G)$, then $\{x, y, z\}$ is independent, contradicting Lemma 8. If $u, v \in A \cap B_1$ and $uv \notin E(G)$, then we can pick $w \in (A' \cap (A_{12} \cup S_2^*)) \setminus \Gamma(z)$ by (11) and (30), so that $\{u, v, z, w\}$ is independent, contradicting $\alpha(G) = 3$. It follows that G_1 is a clique.

Now since $A \cap B_1 \subset L \cup L^*$ by (15), we have $A_{11} \cap (A \cap B_1) = \emptyset$. Thus by (23) and (29), we have $|A_{11} \cup (A \cap B_1)| \geq k$. By (33), we see that every vertex of $L^* \setminus V(G_1)$ has at least k neighbours in G_1.

Similarly, by (14), (20) and (23), we have $|A_{11} \cup (B_1 \cap L^*)| = |A_{11} \cup (B \cap L^*)| \geq k$. Moreover, every vertex of $(A \cap L) \cup M'$ dominates $A_{11} \cup (B_1 \cap L^*)$. Indeed, if $v \in (A \cap L) \cup M'$, then obviously v dominates A_{11} by (6). If $x \in B_1 \cap L^*$ and $xv \notin E(G)$, then we can again apply

[17] $|A \cap B_1| \geq \frac{k+4}{4}$ if (A) holds; $|A \cap B_1| \geq \frac{k+6}{4}$ if (B) holds.

Lemma 9 to x and v if $v \in A \cap L$, and obtain a contradiction to (30); or $\{x, v, z\}$ is independent if $v \in M'$, contradicting Lemma 8. Therefore, every vertex of $((A \cap L) \cup M') \setminus V(G_1)$ also has at least k neighbours in G_1.

Now since $A \subset A_{11} \cup L \cup L^*$ by (17), it follows that $V(G_1) \cup L^* \cup (A \cap L) \cup M' = A \cup A_{11} \cup L^* \cup M' = Q$. We conclude that $G[Q]$ contains the clique G_1 with at least k vertices, while every vertex of $Q \setminus V(G_1)$ has at least k neighbours in G_1. Clearly, $|Q| \geq |G_1| \geq k$. Consequently, if $|Q| = k$, then $Q = G_1$ is a clique on k vertices. Otherwise, if $|Q| > k$, then $G[Q]$ is k-connected. □

Now, recall that $z \in L$ from (31), and $M = \Gamma(z) \cap N \cap S_1^*$ from (32). We let

$$X = \Gamma(z) \cap (A_{12} \cup S_2 \cup S_2^*) \quad \text{and} \quad X' = X \cap S_2. \tag{34}$$

Note that since $M \cup X \subset \Gamma(z) \cap V(H)$ and $z \in L$, we have by (11),

$$|M \cup X| \leq k - 1. \tag{35}$$

We define the subgraph

$$H' = G[(B_1 \cap L^*) \cup M' \cup A_{12} \cup (S_2 \setminus X') \cup S_2^*]. \tag{36}$$

Then note that H' is not k-connected. Otherwise, since $A_{12} \subset V(H) \cap V(H')$ and $|A_{12}| \geq k$ by (8), if H and H' are k-connected, then $H \cup H' = G[(B_1 \cap L^*) \cup N \cup A_{12} \cup S_2 \cup S_2^*]$ would also be k-connected. This contradicts the definition of N in (9), since $B_1 \cap L^* \neq \emptyset$ by (20). We may write

$$V(H') = D \dot{\cup} D' \dot{\cup} T'' \text{ with } |T''| \leq k - 1, \quad |D|, |D'| > 0, \quad E(D, D') = \emptyset. \tag{37}$$

Since $|A_{12}| \geq k$ by (8), we may assume that $D' \cap A_{12} \neq \emptyset$. We now obtain some facts. In view of (6) and $A_{12} \cup (S_2 \setminus X') \cup S_2^* \subset A_{12} \cup S_2 \cup S_2^*$, we have $D \subset (B_1 \cap L^*) \cup M'$. Together with $z \in B_2$ from (31), and the definition of M' in (32), we have

$$\Gamma(z) \cap ((B_1 \cap L^*) \cup M') = \Gamma(z) \cap D = \emptyset. \tag{38}$$

Now, $G[(B_1 \cap L^*) \cup M']$ is a clique, since if we have $x, y \in (B_1 \cap L^*) \cup M'$ and $xy \notin E(G)$, then $\{x, y, z\}$ is independent, contradicting Lemma 8. Consequently, since $D \subset (B_1 \cap L^*) \cup M'$, we have

$$D \subset (B_1 \cap L^*) \cup M' \subset D \cup T''. \tag{39}$$

Finally, since $B_1 \cap L^* \subset T$ by Claim 18, and $M' \subset N \subset T$ by (17) and (32), we have

$$D \subset (B_1 \cap L^*) \cup M' \subset T. \tag{40}$$

Claim 20. *We have*[18]

$$|D| \geq \frac{3(k-1)}{4}. \tag{41}$$

[18] $|D| \geq \frac{3k-4}{4}$ if (A) holds; $|D| \geq \frac{3k-6}{4}$ if (B) holds.

Proof. Recall that $z \in B_2 \subset L$ from (31), and $X = \Gamma(z) \cap (A_{12} \cup S_2 \cup S_2^*)$ from (34). Let

$$Y = (A_{12} \cup S_2^*) \setminus X \subset V(H').$$

We see that $Y \cap ((B_1 \cap L^*) \cup M') = \emptyset$, and Y consists of all non-neighbours of z in $A_{12} \cup S_2^*$. Now, note that every vertex of $(B_1 \cap L^*) \cup M'$ dominates Y. Otherwise, we can find $x \in (B_1 \cap L^*) \cup M'$ and $y \in Y$ with $xy \notin E(G)$. By (38), $\{x, y, z\}$ is independent, contradicting Lemma 8. Consequently from (39), we have $Y \subset T''$.

Now, $A_{11} \subset T'$ by (13); $L^* \subset T' \cup (B \cap L^*) = T' \cup (B_1 \cap L^*)$ by (20); $N \cap S_1^* = M \cup M'$ by (32); and $A_{12} \cup S_2^* \subset X \cup Y$. Together with (39) and $Y \subset T''$,

$$\begin{aligned} A_1 &= A_{11} \cup L^* \cup (N \cap S_1^*) \cup A_{12} \cup S_2^* \\ &\subset T' \cup (B_1 \cap L^*) \cup M \cup M' \cup X \cup Y \\ &\subset T' \cup T'' \cup D \cup (M \cup X). \end{aligned}$$

Therefore, since $|A_2| \leq 2(k-1)$ by Claim 16, and $|M \cup X| \leq k-1$ by (35), we have[19]

$$|D| \geq n - |A_2| - |S| - |T'| - |T''| - |M \cup X| \geq \frac{3(k-1)}{4}.$$

□

Claim 21. *There exists $u \in A' \cap S_2$ with $E(u, D \cup \{z\}) = \emptyset$.*

Proof. Recall that $z \in L$ from (31), and $V(H') = (B_1 \cap L^*) \cup M' \cup A_{12} \cup (S_2 \setminus X') \cup S_2^* = D \dot\cup D' \dot\cup T''$, as defined in (36) and (37). Pick $x \in D \subset (B_1 \cap L^*) \cup M'$ by (39), and let

$$Z = (A' \cap S_2) \setminus (\Gamma(z) \cup \Gamma(x)).$$

Note that from the definition of X' in (34), we have $Z \subset S_2 \setminus X' \subset V(H')$. We claim that $D' \cap Z \neq \emptyset$. Assume that $D' \cap Z = \emptyset$. Since $Z \cap ((B_1 \cap L^*) \cup M') = \emptyset$, and $D \subset (B_1 \cap L^*) \cup M'$ from (39), we have $D \cap Z = \emptyset$, and thus $Z \subset T''$. Again from (39), we have $D' \subset A_{12} \cup (S_2 \setminus X') \cup S_2^* \subset D' \cup T''$. Since $x \in D$, we see that $\Gamma(x) \cap (D' \cup Z) = \emptyset$. Thus

$$e(x, A_{12} \cup (S_2 \setminus X') \cup S_2^*) \leq |T'' \setminus Z|.$$

Also, recall that $A' \subset A_{12} \cup S_2 \cup S_2^*$ from (17). Then note that $A' \setminus Z \subset \Gamma(z) \cup \Gamma(x)$. Otherwise, if we can find $y \in (A' \setminus Z) \setminus (\Gamma(z) \cup \Gamma(x))$, then we have $y \in A' \cap (A_{12} \cup S_2^*)$. Since $xz \notin E(G)$ by (38), we have $\{x, y, z\}$ is independent, contradicting Lemma 8.

Since $Z \subset A'$, we have $|A'| = |A' \setminus Z| + |Z|$. Likewise, $Z \subset T''$ implies $|T''| = |T'' \setminus Z| + |Z|$. Since $|X| \leq |M \cup X| \leq k-1$ by (35), we have

$$\begin{aligned} |A'| &= |A' \setminus Z| + |Z| \\ &\leq |(\Gamma(z) \cup \Gamma(x)) \cap (A_{12} \cup S_2 \cup S_2^*)| + |Z| \\ &\leq |X| + e(x, A_{12} \cup (S_2 \setminus X') \cup S_2^*) + |Z| \\ &\leq |X| + |T'' \setminus Z| + |Z| \\ &= |X| + |T''| \\ &\leq 2(k-1). \end{aligned}$$

[19] $|D| \geq \frac{3k-4}{4}$ if (A) holds; $|D| \geq \frac{3k-6}{4}$ if (B) holds.

Now, since $D \subset (B_1 \cap L^*) \cup M' \subset Q \cap T$ by (32) and (40), we have $|T \setminus Q| \leq |T| - |D|$. Also, in view of (18) and (32), we have $A' \cap Q = \emptyset$, so that $A \subset Q \subset A \dot{\cup} T$, and thus $|A| + |T| = |Q| + |T \setminus Q|$. Together with (4), (41), $|T| \leq k-1$, and $|A'| \leq 2(k-1)$, we have[20]

$$\begin{aligned} |Q| &= n - |A_2 \cup S_3| - |A'| - |T \setminus Q| \geq \frac{2n+1}{3} - |A'| - |T| + |D| \\ &\geq \frac{2n+1}{3} - \frac{9(k-1)}{4} > \frac{n}{3} > k. \end{aligned}$$

By Claim 19, $G[Q]$ is a desired k-connected subgraph, a contradiction.

Therefore, we have $D' \cap Z \neq \emptyset$. Claim 21 now follows by taking $u \in D' \cap Z \subset A' \cap S_2$. $\square$

Let $u \in A' \cap S_2$ be as in Claim 21, and recall that $z \in B_2 \subset L$ from (31). Define

$$S_3'' = S_3 \setminus (\Gamma(z) \cup \Gamma(u)).$$

In view of $z, u \in S \setminus S_3$ and the maximality of S_3, we have $e(z, A_2 \cup (S_3 \setminus S_3'')) \leq k-1$ and $e(u, A_2 \cup (S_3 \setminus S_3'')) \leq k-1$. Moreover, we must have $A_2 \cup (S_3 \setminus S_3'') \subset \Gamma(z) \cup \Gamma(u)$. Otherwise, we can pick $x \in D \subset (B_1 \cap L^*) \cup M'$ by (39), and $w \in A_2 \setminus (\Gamma(z) \cup \Gamma(u))$. By (38), and $E(u, D \cup \{z\}) = \emptyset$ in Claim 21, we have $\{x, z, u, w\}$ is independent, contradicting $\alpha(G) = 3$. It follows that

$$|A_2 \cup (S_3 \setminus S_3'')| \leq 2(k-1). \tag{42}$$

Now, if $S_3'' \neq \emptyset$ and $v \in S_3''$, then v dominates $(A \cap B_1) \cup D$. Otherwise, if there exists $v' \in (A \cap B_1) \cup D$ with $vv' \notin E(G)$, then again by (38) and $E(u, D \cup \{z\}) = \emptyset$, we have $\{z, u, v, v'\}$ is independent, contradicting $\alpha(G) = 3$. Also, note that since $D \subset T$ by (40), we have $(A \cap B_1) \cap D = \emptyset$. Thus, $|(A \cap B_1) \cup D| \geq k$ by (29) and (41). Since $(A \cap B_1) \cup D \subset (A \cap B_1) \cup (B_1 \cap L^*) \cup M' \subset Q$ by (32) and (39), this means that every vertex of S_3'' has at least k neighbours in Q.

Finally, note that since $A' \cap L = \emptyset$ by (18), we have $L \setminus A \subset T$. Moreover, since $D \subset (B_1 \cap L^*) \cup M'$ by (39), we have $D \cap (L \setminus A) = \emptyset$. Therefore, $L \setminus A \subset T \setminus D$. From the definitions of H and Q in (9) and (32), we have

$$\begin{aligned} Q \cup V(H) \cup (L \setminus A) &= A \cup A_{11} \cup L^* \cup M' \cup N \cup A_{12} \cup S_2 \cup S_2^* \cup (L \setminus A) \\ &= A_1 \cup S_1 \cup S_2, \end{aligned}$$

which means that

$$V(G) = (Q \cup S_3'') \cup V(H) \cup (T \setminus D) \cup (A_2 \cup (S_3 \setminus S_3'')).$$

Since $D \subset T$ by (40), we have $|T \setminus D| = |T| - |D|$. Together with (10), (41), (42), and $|T| \leq k-1$, we have[21]

$$\begin{aligned} |Q \cup S_3''| &\geq n - |H| - |T \setminus D| - |A_2 \cup (S_3 \setminus S_3'')| \\ &\geq \frac{2n+1}{3} - |T| + |D| - |A_2 \cup (S_3 \setminus S_3'')| \\ &\geq \frac{2n+1}{3} - \frac{9(k-1)}{4} > \frac{n}{3} > k. \end{aligned}$$

[20] $|Q| \geq \frac{2n+2}{3} - \frac{9k-8}{4} > \frac{n}{3} > k$ if (A) holds; $|Q| \geq \frac{2n+3}{3} - \frac{9k-6}{4} \geq \frac{n}{3} > k$ if (B) holds.

[21] $|Q \cup S_3''| \geq \frac{2n+2}{3} - \frac{9k-8}{4} > \frac{n}{3} > k$ if (A) holds; $|Q \cup S_3''| \geq \frac{2n+3}{3} - \frac{9k-6}{4} \geq \frac{n}{3} > k$ if (B) holds.

By Claim 19, it follows that $G[Q \cup S_3'']$ is a desired k-connected subgraph (whether S_3'' is empty or non-empty), a final contradiction.

This completes Case 2, and the proof of Theorem 6. $\square$

4 Conclusion

In this paper we have shown that, for sufficiently large n, any graph G of order n and independence number α has a k-connected subgraph on at least $\lceil n/\alpha \rceil$ vertices. We presented a construction for $\alpha \geq 4$, which shows that in general we need approximately $n \geq \frac{5\alpha(k-1)}{2}$ to guarantee a k-connected subgraph of order at least $\lceil n/\alpha \rceil$. The determination of the correct lower bound on n in general remains open. We have also determined precisely what "sufficiently large" means in the cases $\alpha = 2$ and $\alpha = 3$. In these cases, our lower bounds on n are accompanied by constructions showing that the bounds are best possible.

Finally, here is a related question. Suppose again that n is not in fact large enough to guarantee a k-connected subgraph on at least $\lceil n/\alpha \rceil$ vertices. What is the largest k-connected subgraph (as a function of α, k and n) that G must nonetheless contain?

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