Hindman's Theorem

For real numbers $x_1, \ldots, x_n$, we define $\mathrm{FS}(x_1, \ldots, x_n)$ to be the set of (nonempty) *finite sums* of the x_i, so that

$$\mathrm{FS}(x_1, \ldots, x_n) = \left\{ \sum_{i \in I} x_i : \emptyset \neq I \subset [n] \right\}.$$

Hindman's theorem states the following

Theorem 1. *Any finite coloring of the natural numbers contains an infinite set A such that $\mathrm{FS}(A)$ is monochromatic.*

We will deduce this from an equivalent set theory version of the result. Let F be the set of finite nonempty subsets of $\mathbb{N}$. For $\emptyset \neq A \subset F$, $f(A)$ denotes the set of finite unions of members of A, excluding the empty union. A set $D \subset F$ is said to be a *disjoint collection* if D is infinite and its members are disjoint.

Theorem 2. *For any finite partition of F into sets $F_1, \ldots, F_n$, there exists an i and a disjoint collection $D \subset F_i$ such that $f(D) \subset F_i$.*

Theorem 1 follows easily from Theorem 2 by identifying a set $A \in F$ with the natural number $\sum_{i \in A} 2^{i-1}$. Specifically, a finite coloring of $\mathbb{N}$ yields a finite coloring of F, which, by Theorem 2, yields a disjoint collection D such that $f(D)$ is monochromatic. Sets in D correspond to natural numbers, and finite unions of sets in D correspond to finite sums of those numbers. (Note that a partition of F into sets $F_1, \ldots, F_n$ is just an n-coloring of F.)

The following definition is the key to the entire proof. If $A \subset F$ and D is a disjoint collection, we say that A is *large* for D if, for every disjoint collection $D' \subset f(D)$, $f(D') \cap A \neq \emptyset$. For example, viewing $\mathbb{N}$ as a disjoint collection of singletons, the set of even-sized subsets of $\mathbb{N}$ is large for $\mathbb{N}$, but the set of odd-sized subsets is not.

The next two lemmas show that a large set "almost" survives partitioning.

Lemma 3. *If A is large for D and $A = A_1 \cup A_2$, there is a disjoint collection $D' \subset f(D)$ such that either A_1 or A_2 is large for D'.*

Proof. Suppose not. Since A_1 is not large for D, there is a disjoint collection $D' \subset f(D)$ such that $f(D') \cap A_1 = \emptyset$. Since A_2 is not large for D', there is a disjoint collection $D'' \subset f(D')$ such that $f(D'') \cap A_2 = \emptyset$. Therefore $f(D'') \cap A = \emptyset$, contradicting the assumption that A is large for D. □

Lemma 4. *Suppose that F is partitioned into sets $F_1, \ldots, F_n$. Then there is some i and a disjoint collection D such that F_i is large for D.*

Proof. Inductively apply Lemma 3. □

Suppose that A is large for D. The goal of the next three lemmas is to show that a certain cleverly-chosen subset of A is still large for some disjoint collection $D' \subset f(D)$.

Lemma 5. *Suppose that A is large for D. Then there is a finite set $E \subset f(D)$, whose members are disjoint, such that for all $d \in f(D)$, if $d \cap (\cup E) = \emptyset$, there is some $e \in f(E)$ with $d \cup e \in A$.*

Proof. Suppose not. Choose $e_1 \in f(D)$ arbitrarily. There is some $e_2 \in f(D)$ with $e_1 \cap e_2 = \emptyset$ and $e_1 \cup e_2 \notin A$. Also, there is some $e_3 \in f(D)$ with $(e_1 \cup e_2) \cap e_3 = \emptyset$ and $e_1 \cup e_3 \notin A, e_2 \cup e_3 \notin A, e_1 \cup e_2 \cup e_3 \notin A$. Continuing in this manner, we obtain disjoint sets $e_1, e_2, e_3, \ldots$ so that if $e \in f(\{e_1, e_2, \ldots, e_n\})$ then $e_{n+1} \cup e \notin A$. Writing $e'_i = e_{2i-1} \cup e_{2i}$ for each i, and setting $D' = \{e'_1, e'_2, e'_3, \ldots\} \subset f(D)$, we see that D' contradicts the assumption that A is large for D. (Note that we don't know that each $e_i \notin A$, so we need to consider the e'_i instead.) □

Lemma 6. *Suppose that A is large for D. Then there is a set $e' \in f(D)$, and a disjoint collection $D' \subset f(D)$, each of whose members is disjoint from e', such that*

$$A(e') = \{a \in A : a \cap e' = \emptyset, a \cup e' \in A\}$$

is large for D'.

Proof. Let $E \subset f(D)$ be as in Lemma 5 and let

$$D_1 = \{d \in D : d \cap e = \emptyset \text{ for all } e \in E\}.$$

Note that $A \cap f(D_1)$ is large for D_1. For every $e \in f(E)$ set

$$A_e = \{a \in A \cap f(D_1) : a \cup e \in A\}.$$

By Lemma 5,

$$A \cap f(D_1) \subset \bigcup_{e \in f(E)} A_e.$$

In other words, we have partitioned the relevant part of A into finitely many ($2^{|E|} - 1$) pieces. Repeated application of Lemma 3 yields a disjoint collection $D' \subset f(D_1)$, and a fixed $e' \in f(E)$, such that $A_{e'}$, and therefore $A(e')$, is large for D'. □

A subtle refinement of Lemma 6 is just what we need to prove the theorem.

Lemma 7. *Suppose that A is large for D. Then there is a set $e'' \in A \cap f(D)$, and a disjoint collection $D'' \subset f(D)$, each of whose members is disjoint from e'', such that*

$$A(e'') = \{a \in A : a \cap e'' = \emptyset, a \cup e'' \in A\}$$

is large for D''.

Proof. By Lemma 6 we have $e_1 \in f(D)$ and $D'_1 \subset f(D)$, each of whose members is disjoint from e'_1, such that

$$A(e'_1) = \{a \in A : a \cap e'_1 = \emptyset, a \cup e'_1 \in A\}$$

is large for D'_1. We next find $e_2 \in f(D'_1)$ and $D'_2 \subset f(D'_1)$, each of whose members is disjoint from e'_2, such that

$$A(e'_2) = \{a \in A(e'_1) : a \cap e'_2 = \emptyset, a \cup e'_2 \in A(e'_1)\}$$

is large for D'_2. Continuing, we find, for each $n \geq 1$, e'_n, D'_n and $A(e'_n)$ with

- $e'_n \in f(D'_{n-1})$
- $D'_n \subset f(D'_{n-1})$
- $d \in D'_n \Rightarrow d \cap e'_n = \emptyset$
- $A(e'_n) = \{a \in A(e'_{n-1}) : a \cap e'_n = \emptyset, a \cup e'_n \in A(e'_{n-1})\}$ large for D'_n.

The family $\{e'_1, e'_2, \ldots\} \subset f(D)$ is itself a disjoint collection, so there are $i_1 < \cdots < i_r$ with

$$e'' = \bigcup_{1 \leq j \leq r} e_{i_j} \in A.$$

Set $D'' = D'_{i_r}$ and we are done. □

Rather remarkably, we now just repeat the proof of Lemma 7 to obtain the full theorem. For let F be partitioned as $F = F_1 \cup \cdots \cup F_n$. By Lemma 4, some F_i is large for some disjoint collection D. We proceed as in the proof of Lemma 7, first finding $e_1'' \in F_i \cap f(D)$ and $D_1'' \subset f(D)$, each of whose members is disjoint from e_1'', such that

$$A(e_1'') = \{a \in F_i : a \cap e_1'' = \emptyset, a \cup e_1'' \in F_i\}$$

is large for D_1''. Continuing, we find, for each $n \geq 1$, e_n'', D_n'' and $A(e_n'')$ with

- $e_n'' \in A(e_{n-1}'') \cap f(D_{n-1}')$
- $D_n'' \subset f(D_{n-1}'')$
- $d \in D_n'' \Rightarrow d \cap e_n'' = \emptyset$
- $A(e_n'') = \{a \in A(e_{n-1}'') : a \cap e_n'' = \emptyset, a \cup e_n'' \in A(e_{n-1}'')\}$ large for D_n''.

Our sought-after disjoint collection is just $\{e_1'', e_2'', e_3'', \ldots\}$.