Rado's Theorem

Suppose that S is a system of equations, with integer coefficients, in variables $x_1, \ldots, x_n$. S is said to be *r-regular* if any r-coloring of $\mathbb{N}$ contains a monochromatic solution to S. S is *regular* if it is r-regular for all r.
For instance, $x + y = z$ is regular - this is Schur's theorem. $x + y = 2z$ is also regular - take $x = y = z$. How about $x + y = 3z$? It turns out that $x + y = nz$ is only regular for $n = 1$ and 2. This and more will follow from Rado's theorem.

Theorem 1. (Rado) *Let $c_1, \ldots, c_n$ be nonzero integers. Then the equation*

$$c_1 x_1 + \cdots + c_n x_n = 0$$

is regular if and only if some nonempty subset of the c_i sums to zero.

We require some preliminaries. The following theorem extends van der Waerden's theorem.

Theorem 2. *For all positive integers k and r, there exists a least integer $n(k, r)$ such that, for any r-coloring of $[n(k, r)]$, there exist integers a and d such that the set $\{a, a+d, \ldots, a+kd\} \cup \{d\}$ (i.e., a $(k+1)$-term progression and its common difference) is monochromatic.*

Proof. We use induction on r. The case $r = 1$ is trivial; $n(k, 1) = k + 1$. I claim that

$$n(k, r) \leq W(kn(k, r - 1) + 1, r).$$

Here is why: suppose we are given an r-coloring of $[N] = [W(kn+1, r)]$, where $n = n(k, r-1)$. We first find a monochromatic (say, red) arithmetic progression $\{a, a+d, \ldots, a+knd\}$. If any of the numbers $d, 2d, \ldots, nd$ is red, we are done. (Specifically, if jd is red, then the set $\{a, a+jd, \ldots, a+jkd\} \cup \{jd\}$ is all red.) If not, the set $\{d, 2d, \ldots, nd\}$ is $r-1$-colored, in which case we are also done - by induction. $\square$

The same proof also yields the following.

Theorem 3. *For all positive integers k, r and s, there exists a least integer $n(k, r, s)$ such that, for any r-coloring of $[n(k, r, s)]$, there exist integers a and d such that the set $\{a, a+d, \ldots, a+kd\} \cup \{sd\}$ (i.e., a $(k+1)$-term progression and s times its common difference) is monochromatic.*

Next, we prove a special case of Rado's theorem.

Lemma 4. *For all nonzero integers s and t, the equation $sx + ty = sz$ is regular.*

Proof. We may assume that $s > 0$. By possibly interchanging the roles of x and z, we may also assume that $t > 0$. By the previous theorem, given any r-coloring of $\mathbb{N}$, we can find a and d such that $\{a, a+d, \ldots, a+td\} \cup \{sd\}$ is monochromatic. Now just take $x = a, y = sd, z = a + td$. □

Proof of Rado's theorem. First we prove that the condition is sufficient. Suppose that, without loss of generality, $c_1 + \cdots + c_k = 0$. If $k = n$, we can just take $x_1 = \cdots = x_n = 1$. Suppose then that $k < n$. We set

$$x_i = \begin{cases} x & \text{for } i = 1 \\ z & \text{for } 2 \leq i \leq k \\ y & \text{for } i \geq k+1. \end{cases}$$

The equation now reduces to

$$c_1 x + (c_2 + \cdots + c_k) z + (c_{k+1} + \cdots + c_n) y = 0,$$

or, writing $s = c_1, t = c_{k+1} + \cdots + c_n$, and recalling that $c_1 + \cdots + c_k = 0$,

$$sx + ty = sz,$$

which is regular by the lemma.

Next we prove that the condition is necessary. Choose a prime p satisfying $p > \sum |c_i|$. Color each natural number by the last nonzero digit in its base p expansion; this yields a $(p-1)$-coloring of $\mathbb{N}$. For instance, if $p = 5$, we color the numbers 100, 101 and 105 with colors 4, 1 and 1 respectively. Suppose now that $x_1, \ldots, x_n$ is a monochromatic solution to $\sum c_i x_i = 0$. Let l be the largest integer such that $p^l | x_i$ for all i. By dividing through by p^l if necessary (note that this does not change the color of the x_i), we may assume that $l = 0$; further, we may assume that $x_1 \equiv \cdots \equiv x_k \equiv d \pmod{p}$ and $x_{k+1} \equiv \cdots \equiv x_n \equiv 0 \pmod{p}$, where $1 \leq k \leq n$. Consequently,

$$0 \equiv c_1 x_1 + \cdots + c_n x_n \equiv d(c_1 + \cdots + c_k) \pmod{p},$$

and so

$$c_1 + \cdots + c_k \equiv 0 \pmod{p},$$

which is a contradiction, since $p > \sum |c_i|$.

Rado's theorem has a generalization to systems of equations. Suppose that S is a system of m equations (with integer coefficients) in n variables $x_1, \ldots, x_n$. S can be written as $C\mathbf{x} = \mathbf{0}$ in the usual way, or as

$$x_1\mathbf{c_1} + \cdots + x_n\mathbf{c_n} = \mathbf{0},$$

where the $\mathbf{c_i}$ are the columns of C. Then S (or, equivalently, C) is said to satisfy the *columns condition* if, for some partition $[n] = B_1 \cup \cdots \cup B_t$,

- $\sum_{i \in B_1} \mathbf{c_i} = \mathbf{0}$
- $\sum_{i \in B_s} \mathbf{c_i} \in \text{span}\{\mathbf{c_i} : i \in B_1 \cup \cdots \cup B_{s-1}\}$ for $2 \le s \le t$.

Theorem 5. (Rado) $C\mathbf{x} = \mathbf{0}$ *is regular if and only if* C *satisfies the columns condition.*

The proof follows similar lines to the case $m = 1$ discussed above. The base p colorings just defined force the necessity of the columns condition. The sufficiency is proved by finding a monochromatic solution inside an appropriate "(m, p, c)-set", which is a generalization of the "arithmetic progression plus common difference" structure featured above.