Van der Waerden's Theorem

Van der Waerden's theorem states the following.

Theorem 1. *A finite coloring of* $\mathbb{N}$ *contains arbitrarily long monochromatic arithmetic progressions.*

We will in fact prove a seemingly stronger theorem (which is actually equivalent to the theorem above, by "compactness").

Theorem 2. *For all positive integers* k *and* r*, there exists a least integer* $W(k,r)$ *such that any* r*-coloring of* $[W(k,r)]$ *contains a* k*-term monochromatic arithmetic progression.*

Proof. The key to the proof is the idea of *color-focusing*. Suppose $A_1, A_2, \ldots, A_s$ are disjoint arithmetic progressions of length $k-1$, where $A_i = \{a_i, a_i + d_i, \ldots, a_i + (k-2)d_i\}$. The A_i are said to be *focused* at f if, for all i, $a_i + (k-1)d_i = f$. In other words, f is the "missing" kth term of each of these progressions. Further, if, in some coloring, each A_i is monochromatic, with each A_i receiving a different color, then the progressions together are said to be *color-focused* at f. The point is that, when $s = r$, an r-coloring (of $\mathbb{N}$, say) containing a set of r color-focused arithmetic progressions $A_1, A_2, \ldots, A_r$, each of length $k-1$, must contain a monochromatic arithmetic progression of length k. This is because the common focus f of the A_i must receive one of the r colors, whereupon it extends one of the $(k-1)$-term monochromatic progressions to length k.
From now on, we will write AP for arithmetic progression, and MAP for monochromatic arithmetic progression.
Another important idea in this proof is the use of *double induction*. The main "outer" induction is on k. But, for fixed k and a given r, we will establish the finiteness of $W(k,r)$ using an inductive argument in which we'll assume the finiteness of certain numbers $W(k-1,t)$, where t will typically be much, much larger than r. Also, for fixed r, our inductive step will also use induction (but on a new variable s, related to the discussion above).

Now to the proof itself. By the pigeonhole principle, $W(2,r) = r+1$ for all r. Next, suppose we know that $W(k-1,t)$ is finite *for all* t. Our aim is to show that, for a fixed r, $W(k,r)$ is also finite.
To do this, we will show, for each $s \leq r$, the existence of a number $V(k,r,s)$ such that any r-coloring of $[V(k,r,s)]$ contains either

- A MAP of length k, or
- A set $A_1, A_2, \ldots, A_s$ of color-focused $(k-1)$-term MAPs, together with their common focus.

The case $s = 1$ is trivial: just take $V(k,r,1) = 2W(k-1,r)$. Assume that we know that $V(k,r,s-1)$ is finite. I claim that

$$V(k,r,s) \leq 2V(k,r,s-1)W(k-1, r^{V(k,r,s-1)}).$$

Here is why: suppose we are given an r-coloring of $[N]$, where $N = 2V(k,r,s-1)W(k-1, r^{V(k,r,s-1)})$. We break the coloring up into $2W = 2W(k-1, r^{V(k,r,s-1)})$ "blocks" of length $V = V(k,r,s-1)$. There are r^V ways to color each block, so, by construction (this is the induction on k), there is a progression of identically colored blocks $B_l, B_{l+m}, \ldots, B_{l+(k-2)m}$ of length $k-1$ among the first W blocks, whose kth term is also among the $2W$ blocks colored.
Now we look inside each (identically colored) block B_{l+jm}. By hypothesis (this is the induction on s), we can find $s-1$ color-focused progressions of length $k-1$, together with their focus, within each such block. Suppose that, in color i (where $1 \leq i \leq s-1$) and in block $l+jm$ (where $0 \leq j \leq k-2$), the progression is:

$$\{a_i + jmV, a_i + d_i + jmV, \ldots, a_i + (k-2)d_i + jmV\}, \text{ with focus } f + jmV.$$

Unless we have a monochromatic k-term progression, all the foci $f + jmV$ (where $0 \leq j \leq k-2$) are colored with a new color: s, say. Finally, writing

$$A_i = \begin{cases} \{a_i, a_i + (d_i + mV), a_i + 2(d_i + mV), \ldots, a_i + (k-2)(d_i + mV)\} & 1 \leq i \leq s-1 \\ \{f, f + mV, f + 2mV, \ldots, f + (k-2)mV\} & i = s, \end{cases}$$

we observe that $A_1, A_2, \ldots, A_s$ form a set of s color-focused progressions of length $k-1$, with common focus $f + (k-1)mV \leq N$.
This completes the ("inner") induction on s. For the "outer" induction on k, note that, by the argument at the start of this proof, we must have $W(k,r) \leq V(k,r,r)$.

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