Bootstrap Percolation in Random Geometric Graphs

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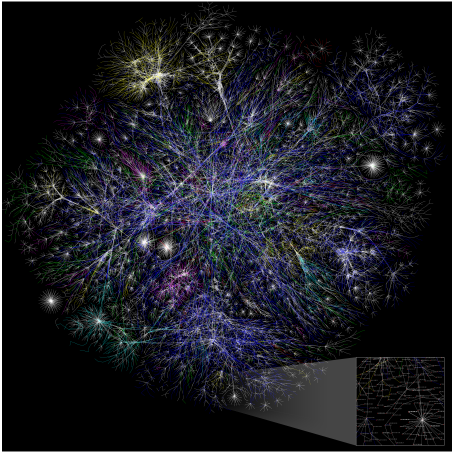
Joint work with Victor Falgas-Ravry (Umeå University)

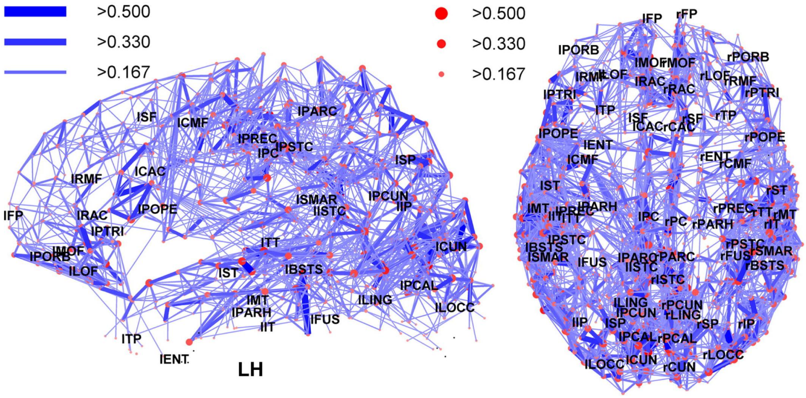
Large networks are everywhere.



Internet

Credit: Matt Britt

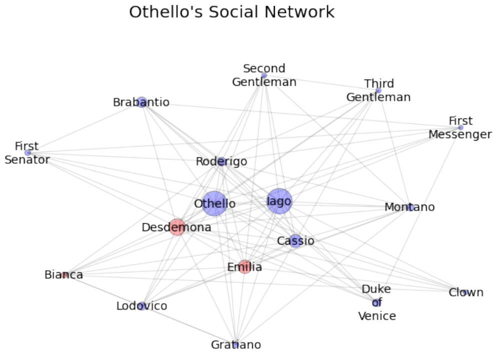
Another example:



Brain

Credit: Hagmann P, Cammoun L, Gigandet X, Meuli R, Honey CJ, Wedeen VJ, Sporns O

And another:



Credit: Shakespeare, Adam Palay

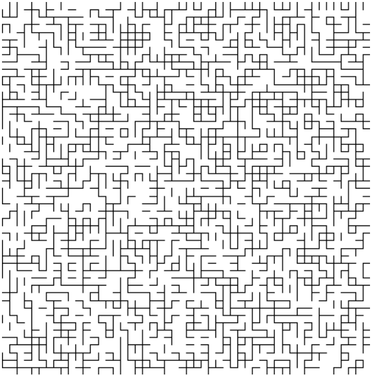
Phase transitions are also everywhere, especially in physics.

- Water boils at $100^\circ C$.
- The Curie temperature of iron is $770^\circ C$.
- An egg scrambles at $70^\circ C$.

Why?

A phase transition (math: *sharp threshold*) often has a large random network underneath it.

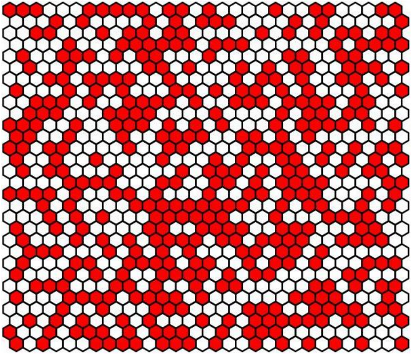
Bond percolation in the square lattice (Broadbent and Hammersley 1957)



Edges (bonds) included independently with probability p

Kesten (1980) $p_c = \frac{1}{2}$

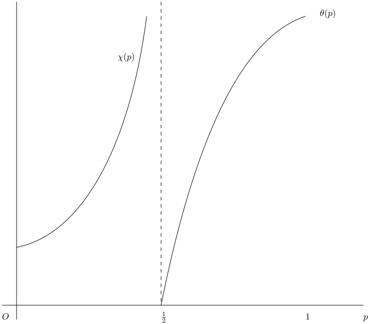
Face percolation in the hexagonal lattice



Kesten (1982) $p_c = \frac{1}{2}$
Schramm, Smirnov, Lawler/Schramm/Werner, Smirnov/Werner (early 2000s)

$$\beta = \frac{5}{36},\ \gamma = \frac{43}{18},\ \delta = \frac{91}{5},\ \eta = \frac{5}{24},\ \nu = \frac{4}{3}$$

β and γ

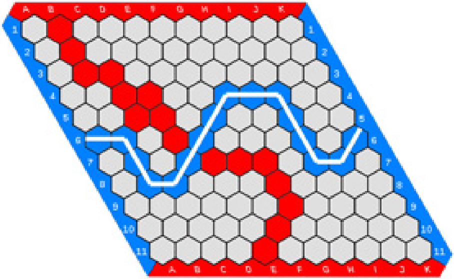


$C =$ connected cluster containing the origin

$\theta(p) = \mathbb{P}_p(|C| = \infty) = (p - p_c)^{\beta + o(1)}$ as $p \downarrow p_c$

$\chi(p) = \mathbb{E}_p(|C|) = (p - p_c)^{-\gamma + o(1)}$ as $p \uparrow p_c$

$p_c = \frac{1}{2}$



$\mathbb{P}_{1/2}(R) = \mathbb{P}_{1/2}(B)$

$\mathbb{P}_{1/2}(R) + \mathbb{P}_{1/2}(B) = 1$

$\mathbb{P}_{1/2}(R) = \mathbb{P}_{1/2}(B) = \frac{1}{2}$

Other models exhibiting phase transitions:

random graphs
branching processes

Poisson processes - definition

• Tessellate $\mathbb{R}^2$ with unit squares

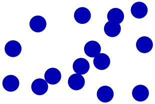
• In each square S_i, independently, place X_i points uniformly at random, where $X_i \sim \mathrm{Po}(1)$, i.e.,

$$\mathbb{P}(X_i = k) = \frac{1}{ek!}$$

• This has many more nice properties than one might expect

[A Poisson process is a limit as $N \to \infty$ of the process obtained by placing N points uniformly at random in a box of area N.]

Poisson processes - properties



$\mathcal{P} =$ **Poisson process** of intensity 1 in $\mathbb{R}^2$

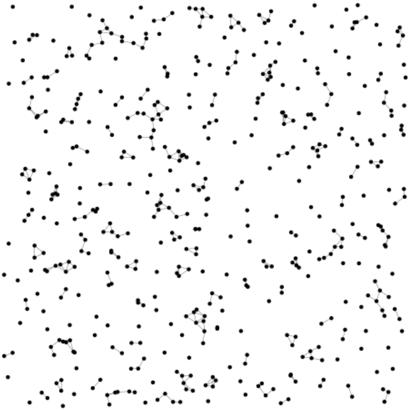
• Number of points X inside any region A is a random variable with the Poisson distribution of mean $|A|$, so that

$$\mathbb{P}(X = k) = \frac{e^{-|A|}|A|^k}{k!}$$

• Disjoint regions are independent

[A Poisson process is a limit as $N \to \infty$ of the process obtained by placing N points uniformly at random in a box of area N.]

Random geometric graphs (Gilbert 1961)

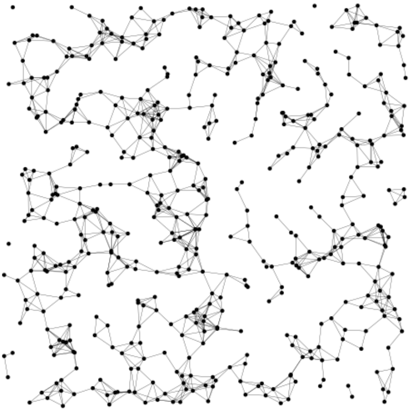


Vertices (nodes) are a Poisson process of intensity 1

Edges join vertices at distance less than r

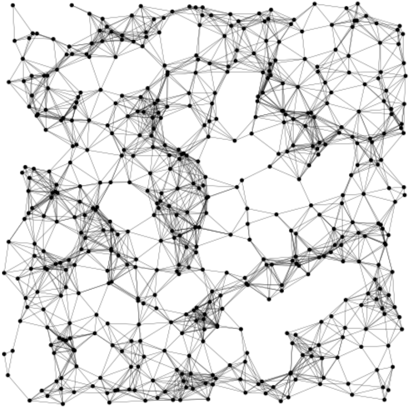
Gilbert's motivation: communications networks

Percolation



Hall (1985) $0.833 < r_{\mathrm{perc}} < 1.836$
Balister, Bollobás and Walters (2005) $1.1978 < r_{\mathrm{perc}} < 1.1989$
- semi-rigorous, high confidence result

Connectivity



Penrose (1997) $\pi r_{\mathrm{conn}}^2(n) = \log n$
Obstruction to connectivity: isolated vertices
At the threshold, $\mathbb{E}(\textit{isolated vertices}) = 1$

Disclaimers

Instead of using a square, we put the points in a **torus**, to avoid boundary effects.

Connectivity is only guaranteed **with high probability**, i.e., with probability tending to 1 as $n \to \infty$.

The same applies to (almost) every other definitive-sounding statement I'll make.

The Bradonjić-Saniee model (2014)

Start with the Gilbert model, above the connectivity threshold

$$\pi r^2 = a \log n \text{ with } a > 1$$

Initially infect vertices independently with probability p: this is A_0
Each vertex expects

$a \log n$ neighbors

$ap \log n$ infected neighbors

$A_t :=$ set of infected vertices at time t
In each discrete time step ($t = 1, 2, \ldots$)

For each $v \notin A_t$ (i.e. each uninfected v)

If v has at least $a\theta \log n$ infected neighbors

- v becomes infected (and stays infected forever)

Repeat for each vertex v to get A_{t+1}

Repeat for each t to get A_∞

What proportion $|A_\infty|/n$ of the graph eventually becomes infected?

Theorem (Bradonjić and Saniee 2014)

For $x > 0$, define

$$J(x) = \log x - 1 - 1/x$$

and write J_R^{-1} for the inverse of J on $[1, \infty]$. Then if

$$p < p' = \theta / J_r^{-1}(1/a\theta)$$

then no initially uninfected vertex becomes infected.

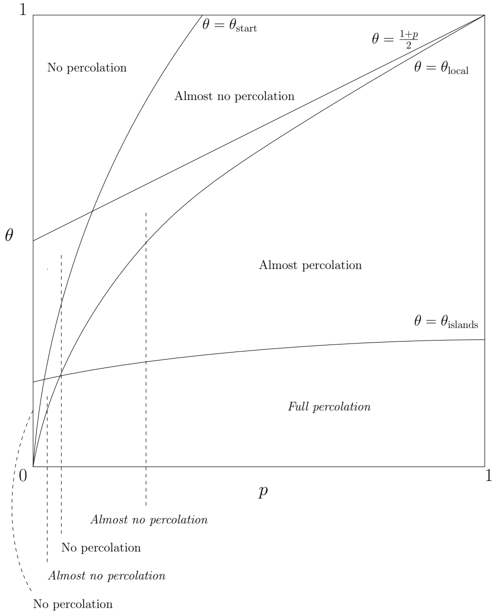
Theorem (Bradonjić and Saniee 2014)

If

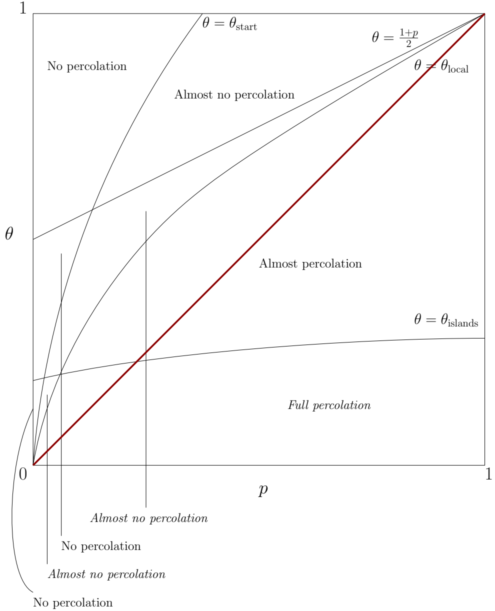
$$p > p'' = \min\left\{\theta, \frac{5\pi\theta}{J_r^{-1}(1/a\theta)}\right\}$$

then every initially uninfected vertex becomes infected.

Theorem (Falgas-Ravry and S 2022+)



Basic orientation - the threshold $\theta = p$



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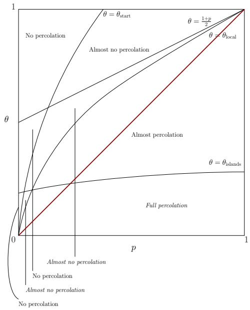
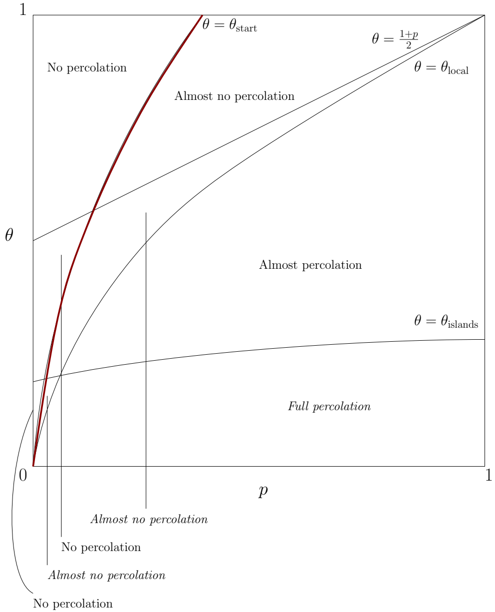


If $\theta < p$, almost everything becomes infected immediately.

If $\theta > p$, almost no new infections occur initially.

But this turns out to be completely irrelevant.

The starting threshold $\theta = \theta_{\text{start}}(p)$



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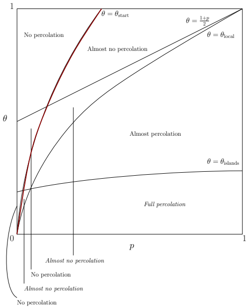
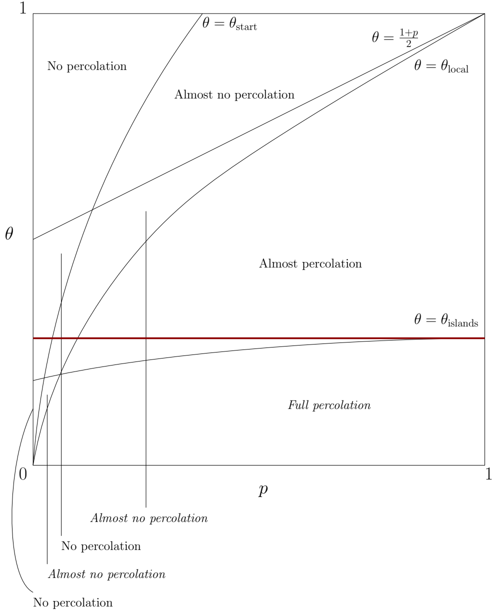


Sometimes, even when the threshold θ is much greater than p, some uninfected vertices will see $a\theta \log n$ infected neighbors, despite only expecting to see only $ap \log n$.
This will happen when

$$f_{\text{start}}(a, p, \theta) = a(p - \theta + \theta \log(\theta/p)) < 1.$$

In this case, the infection will start to spread, and grow to at least logarithmic size.

The simple stopping threshold $\theta = \theta_{\text{stop}}$



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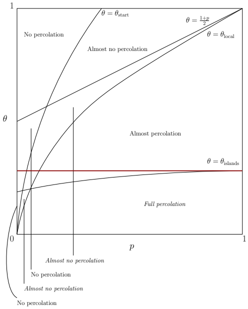
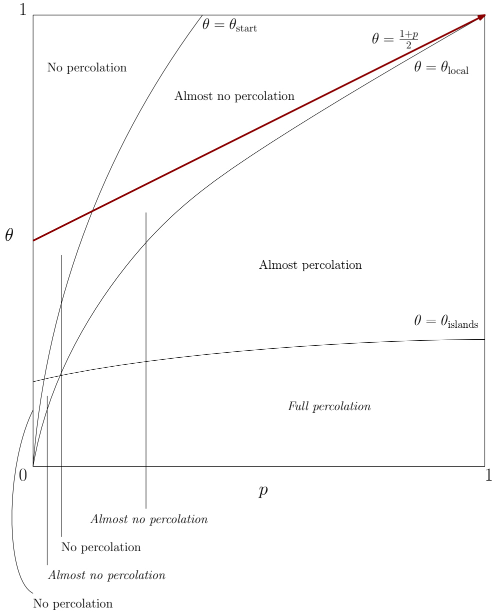


On the other hand, some initially uninfected vertices will not even have $a\theta \log n$ neighbors, despite only expecting to see $a \log n$. These vertices can never become infected.
This will happen when

$$f_{\text{stop}}(a, \theta) = a(1 - \theta + \theta \log \theta) < 1.$$

This yields a simple necessary condition for full percolation.

The growing threshold $\theta = \frac{1+p}{2}$



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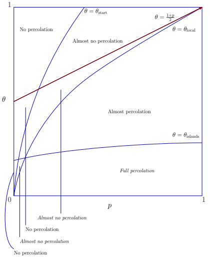
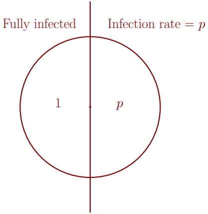


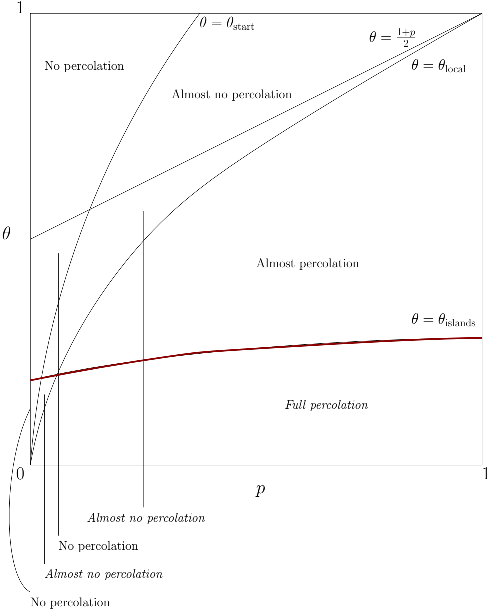


When infections have broken the logarithmic barrier, they will grow as long as

$$\theta < \frac{1+p}{2}$$

But they need to clear a lot of local hurdles first.

The threshold for full percolation $\theta = \theta_{\text{islands}}(p)$



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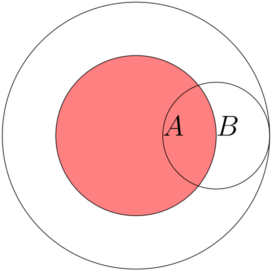

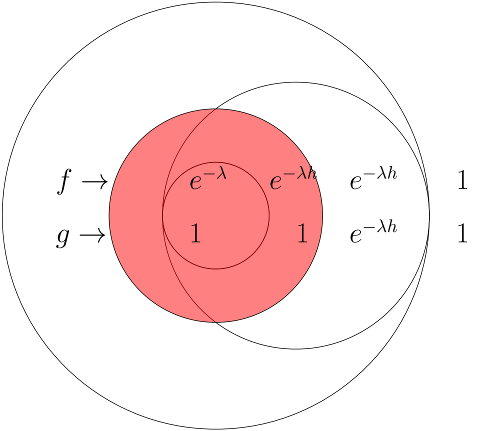

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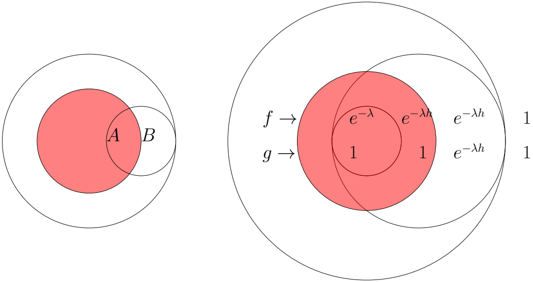
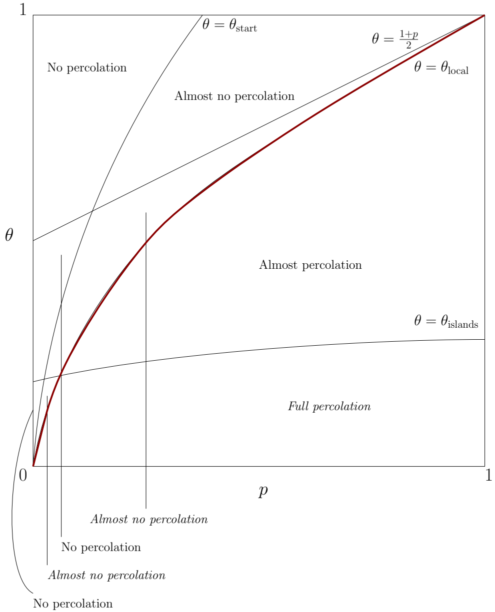


At the other end, there are small obstructions to full percolation, of radius τr. These can be optimized (for fixed τ) using Lagrange multipliers, and the Euler-Lagrange equations

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{\partial F}{\partial y}$$

We then optimize over τ, to find the most likely obstruction.

The local growth threshold $\theta = \theta_{\mathrm{local}}(p)$



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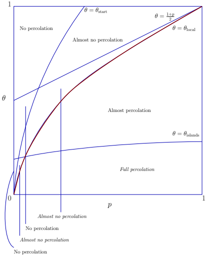
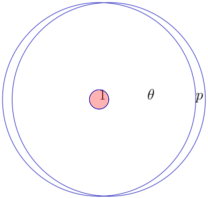


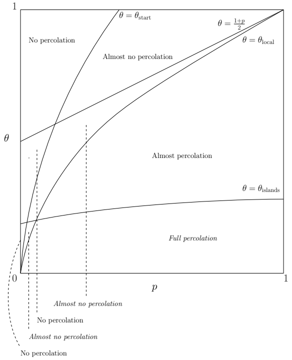


To break the logarithmic barrier, infections need to do more than just start.

They need to be able to expand beyond each radius τr.

This yields a Lagrange multiplier problem with **infinitely many conditions**.

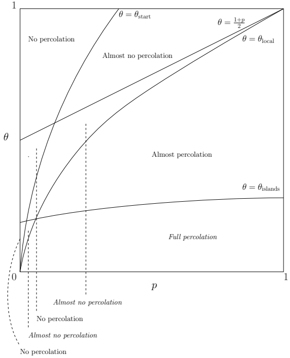
What did we actually prove?



The local growth threshold is only a sufficient condition for local growth, and the islands threshold is only a necessary condition for full percolation.

Accordingly, these thresholds only provide a lower bound for local growth and an upper bound for full percolation.

What did we actually prove?

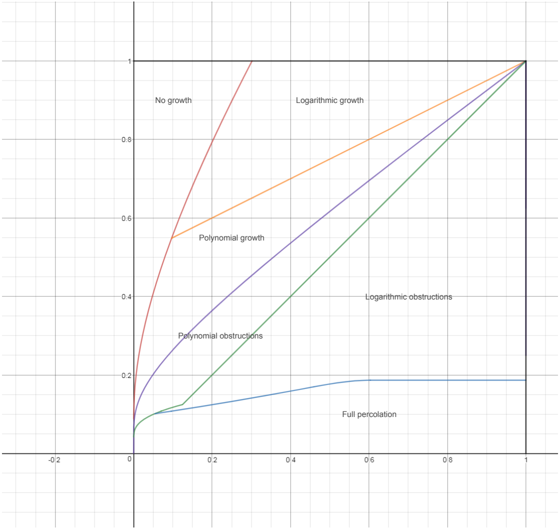


We did prove that the growing condition is the true threshold for local infections becoming global.

Some tools in the proof:

tessellation arguments (fine and rough tilings)
discrete isoperimetric inequalities

It's more complicated in one dimension



Come to Bellingham in April!

Thank you for your attention!