Continued Fractions

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You probably know that $\frac{22}{7}$ is a good approximation to π. *How* good is it actually? We can measure this by looking at the absolute value of the difference between $\frac{22}{7}$ and π, or, in symbols,

$$\left|\pi - \frac{22}{7}\right|$$

and comparing this to $\frac{1}{14}$. Why $\frac{1}{14}$? Because *any* real number x, rational or irrational, can be approximated by some fraction of the form $\frac{p}{7}$, where p is an integer, and where

$$\left|x - \frac{p}{7}\right| \leq \frac{1}{14}.$$

Make sure you understand why this is, before reading further.

It turns out that

$$\frac{\left|\pi - \frac{22}{7}\right|}{\frac{1}{14}} \approx 0.0177,$$

which is *small*, indicating that the approximation is indeed pretty good. What if we wanted to find an even better approximation? Let's examine where $\frac{22}{7}$ came from in the first place. The greatest integer less than π is 3, and subtracting 3 from π we get $0.14159\ldots$. Now the reciprocal of this is just a bit more than 7, so $\pi \approx 3 + \frac{1}{7} = \frac{22}{7}$. The point is that we can *continue this process*. If we subtract 7 from $\frac{1}{\pi-3} = 7.06251\ldots$, and take the reciprocal of *that*, we get something just very slightly less than 16. So

$$3 + \frac{1}{7 + \frac{1}{16}} = \frac{355}{113}$$

is going to be an even better approximation. In fact this *excellent* approximation – $|\pi - \frac{355}{113}| \approx 0.000000267$ and $0.000000267/\frac{1}{226} \approx 0.00006$ – was known to the Chinese astronomer Tsu Ch'ung-Chih over 1500 years ago.

We could continue expanding this fraction as

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{16 - \cfrac{1}{\cdots}}}$$

but it is usual to always have + signs, so that we would actually continue as follows:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}$$

– the 292 providing additional confirmation of the excellence of Tsu Ch'ung Chih.

Now it turns out that stopping the expansion (which is called the *continued fraction expansion*) of π at any point always yields very good rational approximations to π. Let's try to figure out why. It should be obvious to you that π differs from $\frac{22}{7}$ by less than $\frac{1}{7}$. Can you see why the error is actually guaranteed to be less than $\frac{1}{56}$? This is the key to understanding why the so called *continued fraction convergents* provide such good approximations.

There are many interesting connections between continued fractions and other topics in mathematics. For instance, there is nothing stopping us from computing the continued fraction expansion of a *rational* number instead of an irrational one – nothing stopping us, that is, until the expansion itself stops. Here is the continued fraction for $\frac{15}{64}$:

$$\frac{15}{64} = \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{3}}}}.$$

There is a connection with the Euclidean algorithm here – see if you can discover it.

Here's another example which arose in some situation connected with GPS when I was working at Philips Research Laboratories 10 years ago (and doubtless in many other contexts as well). I have actually forgotten the details of the GPS context, but anyway. Take an irrational number θ, and consider the sequence of fractional parts of multiples of θ, i.e. $\{\theta\}, \{2\theta\}, \{3\theta\}, \ldots$. We can visualize this process by imagining walking around a circle of circumference 1 with footsteps of length θ, and recording where our feet land. Next, we divide the circle into two equal halves, and, after some large number of steps, we count the number of footsteps in each half. We would like these two numbers to be about the same. Otherwise there is said to be "bias". Choosing values of θ which avoid bias of this type is facilitated by calculating the continued fraction expansion of θ.

A third example is the connection with *Pell's equation*

$$x^2 - dy^2 = 1$$

where d is some squarefree positive integer and x and $y \neq 0$ are required to be integers. For a typical solution, x and y will be large, and so we will have

$$\frac{x^2}{y^2} = d + \frac{1}{y^2} \approx d$$

and hence

$$\frac{x}{y} \approx \sqrt{d}.$$

It turns out that x and y can be obtained as the numerators and denominators of some of the convergents for $\sqrt{d}$!

Reading about the history of mathematics, it seems that continued fractions used to play a more central role at the frontier. For example, the first proof that π is irrational, given by Johann Lambert in 1760, used continued fractions. Not much is known about the continued fraction expansion of π, but amazingly there is one for $\tan x$:

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \frac{x^2}{\cdots}}}}}.$$

Lambert argued that if $x \neq 0$ is rational, then, using this expansion, $\tan x$ cannot be. But $\tan \frac{\pi}{4} = 1$ *is* rational, so $\frac{\pi}{4}$, and hence π itself, cannot be.

Homework

1. Calculate the continued fraction expansion of the golden ratio.

2. (Putnam 1995) Evaluate

$$\left\{ 2207 - \cfrac{1}{2207 - \frac{1}{2207 - \cdots}} \right\}^{1/8}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.
[**Hint.** Solve $x = 2207 - \frac{1}{x}$.]