Telescoping sums and products

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The basic method

The simplest example of a telescoping sum is perhaps

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

Recall that this sum is defined to be the limit as $N \to \infty$ of

$$\sum_{n=1}^{N} \frac{1}{n(n+1)}$$

Looking at the first few partial sums we might see that they seem to be approaching 1, but how are we supposed to prove this? The key is to write each term as follows:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

so that

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots \\ &= \end{aligned}$$

In general one has to be a bit careful with rearranging infinite series, but in this case (and usually, in the Putnam) we are OK, since the above reasoning in fact shows that

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$$

and then we can just take the limit. From now on, I won't worry about things like this.

In the above example, part of each term canceled with part of the next one. But the cancelation doesn't have to occur among consecutive terms – we could also skip a few. Here is an example illustrating this. Suppose we have to evaluate

$$S = \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

As before, we can express each term as a difference:

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

and so

$$\begin{aligned} S = \sum_{n=1}^{\infty} \frac{1}{n(n+3)} &= \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right) \\ &= \frac{1}{3}\left\{\left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \cdots\right\} \\ &= \frac{1}{3}\left\{1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \cdots\right\} \end{aligned}$$

but this time we have to *wait a bit* for the terms to cancel, and some terms (which?) don't cancel at all. You can finish the problem yourself!

If I had to summarize the method, I would say that we express each term $f(n)$ as a difference $g(n) - g(n+1)$. Then, as long as $g(n) \to 0$, we have

$$\sum_{n=1}^{\infty} f(n) = \lim_{N\to\infty} \sum_{n=1}^{N} f(n) = \lim_{N\to\infty} \sum_{n=1}^{N} \{g(n) - g(n+1)\} = \lim_{N\to\infty} \{g(1) - g(N+1)\} = g(1)$$

but, as above, there is also a version of this based on writing $f(n) = g(n) - g(n+k)$ for some integer k.

Infinite products

These are basically the same. For example:

1. (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$$

If we can express each term in the form

$$\frac{n^3-1}{n^3+1} = \frac{g(n)}{g(n+1)}$$

or something similar, then part of each term will cancel with part of the next, exactly as above, except that the cancelation is multiplicative rather than additive. The infinite product itself is defined as the limit of partial products in the obvious way.

Tricks

Once you have decided to use this method, you have to figure out how to write each term $f(n)$ as a difference $g(n) - g(n+k)$. Often, you can use partial fractions, but sometimes you need to have some tricks up your sleeve. For instance, suppose you have to evaluate

$$\sum_{n=1}^{N} \frac{1}{\sqrt{n}+\sqrt{n+1}}$$

Rationalizing the denominator should be second nature to you, and after doing this you/we find that the series telescopes.

Here's a more complicated example:

$$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^3}{(2k+1)^4+4}$$

In this case, you can use the identity

$$x^4+4 = (x^2-2x+2)(x^2+2x+2)$$

which you could probably work out, *if you knew that x^4+4 can be factored at all.* Then you can decompose each term into partial fractions, but quite a lot of ingenuity is still required in order to complete the solution.

Homework

1. Finish all the problems discussed above.

2. Let $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$$

3. (Putnam 1977) For $0 < x < 1$, express

$$\sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}}$$

as a rational function of x.

4. (Putnam 1978) Express

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2 n + m n^2 + 2mn}$$

as a rational number.

5. (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

6. (Putnam 1986) Evaluate

$$\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2 + n + 1)$$

where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.