

# Integrals

October 31, 2008

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I'll illustrate some tricks using some

## Examples

1. (Putnam 1946) If  $a(x), b(x), c(x)$  and  $d(x)$  are polynomials in  $x$ , show that

$$\int_1^x a(t)c(t) dt \cdot \int_1^x b(t)d(t) dt - \int_1^x a(t)d(t) dt \cdot \int_1^x b(t)c(t) dt$$

is divisible by  $(x - 1)^4$ .

At first sight, you might plan on somehow evaluating the integrals and then dividing by  $(x - 1)^4$ . But obviously you can't do that because you don't know  $a(x), b(x), c(x)$  or  $d(x)$ . In fact, there are very few things you *can* do with the horrible-looking expression in the question. Is there anything we can do with it at all? Well, we can give it a name:  $F(x)$ . Now, what is  $F(1)$ ? Clearly it is 0. So  $F(1) = 0$ . Also,  $F(x)$  is a polynomial. So,  $F(x)$  is a polynomial and  $F(1) = 0$ . Therefore,  $F(x)$  is divisible by  $(x - 1)$ . Now what? Well, if we can show that  $F'(1) = 0$  then we will have shown that  $F(x)$  is divisible by  $(x - 1)^2$ . But this too is not so hard. Now we are well on the way to solving the problem completely.

2. (Putnam 1953) Find

$$\int_0^{\pi/2} \ln \sin x dx.$$

Actually, this question had a hint, which I've removed, thus making it harder. The key here is to try and exploit the symmetry between sin and cos. At present there is no cos and no symmetry, so how can we create some?

3. (Putnam 1968) Prove that

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

This is totally standard: I could set it in a Math 125 test (and then my teaching evaluation results would be terrible).

4. (Putnam 1980) Evaluate

$$\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$

It's always good idea with trigonometric integrals to try the substitution  $x = \pi/2 - y$ . By the way, if you ever see  $1 + x^2$  in an integral, substituting  $x = \tan \theta$  is not a bad idea. If you see  $\sqrt{1-x^2}$ , substitute  $x = \sin \theta$ . Another useful substitution for trigonometric integrals is  $t = \tan(x/2)$ , because then  $\sin x$ ,  $\cos x$  and  $\tan x$  are all nice functions of  $t$ . (Exercise: work out the details.)

5. (Putnam 1987) Compute

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} dx.$$

Basic exam technique: the integral is a definite integral for a reason – almost certainly there is no closed-form solution for the indefinite version. So don't try finding one. Also, it's always a good idea to try and tidy up the integrand so that it looks as symmetrical as possible. Here, we do this by observing that

$$9 - x = 6 + \text{something}$$

and

$$x + 3 = 6 - \text{the same thing}$$

Working out the details,  $9 - x = 6 + (3 - x)$  and  $x + 3 = 6 - (3 - x)$ . So, writing  $y = 3 - x$ , the integral becomes

$$\int_{-1}^1 \frac{\sqrt{\ln(6+y)}}{\sqrt{\ln(6+y)} + \sqrt{\ln(6-y)}} dy.$$

So far, all we've done is *get ready to solve* the problem, but look how much nicer the integrand looks now. In fact, it even suggests how to solve the problem completely.

### Homework

1. (Putnam 1944) Assuming that  $f(x)$  is continuous on  $[0, 1]$ , show that

$$6 \int_{x=0}^{x=1} \int_{y=x}^{y=1} \int_{z=x}^{z=y} f(x)f(y)f(z) \, dx \, dy \, dz = \left( \int_0^1 f(t) \, dt \right)^3 .$$

2. (Putnam 1965) Evaluate

$$\lim_{n \rightarrow \infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2 \left( \frac{\pi}{2n} (x_1 + x_2 + \cdots + x_n) \right) dx_1 dx_2 \cdots dx_n .$$