Discrete functional equations

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Problems involving integer-valued functions on the integers occur often in the Putnam. There is no general method for solving them, but the best advice I can give you is to *experiment*. If the question involves an integer-valued function f, defined on the integers (or positive integers), try to calculate as many values $f(n)$ as you can. Try to determine which values of f are crucial to finding other values. If you cannot find, say, $f(1)$, perhaps you can write down an equation or an inequality that it must satisfy. Finally, even before you have calculated any values of f, maybe you can guess a formula for it based on the information in the question.

Examples

1. (Putnam 1983) Let $f(n) = n + \lfloor \sqrt{n} \rfloor$ where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Prove that, for every positive integer m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

contains at least one square of an integer.

I'll write, for instance, $5 \to 7$ to indicate that $f(5) = 7$. Let's choose some reasonably large (and hence representative) values of m and compute $f(m), f(f(m)), \ldots$ until we "hit" a square. Maybe we will spot a pattern. If we take $m = 100$ or $m = 121$, these are already square, so how about trying $m = 101, 102, \ldots, 120$. We see that $101 \to 111 \to 121; 102 \to 112 \to 122 \to 133 \to 144; 103 \to 113 \to 123 \to 134 \to 145 \to 157 \to 169; 104 \to 114 \to 124 \to 135 \to 146 \to 158 \to 170 \to 183 \to 196$. Now you should spot a pattern.

2. (Putnam 1992, modified) Find the integer-valued function f defined on the integers that satisfies $f(f(n)) = n$ and $f(f(n+2)+2) = n$ for all integers n, and $f(0) = 1$.

Can you find $f(1)$? How about $f(3)$? Now, can you determine any more values of f? Why not draw a graph showing all the values of f that you can calculate?

3. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n),$$

for all $n \geq 1$.

4. (Putnam 1963) Let $\{f(n)\}$ be a strictly increasing sequence of positive integers such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ for every relatively prime pair of positive integers m and n (the greatest common divisor of m and n is equal to 1). Prove that $f(n) = n$ for every positive integer n.

Homework

The *International Mathematical Olympiad* (IMO) is an annual mathematics competition for high school students. This year it was held in Madrid: 97 countries took part, most sending a team of 6 students. Here are some problems from past IMOs.

1. (IMO 1978) The set of all positive integers is the union of two disjoint subsets

$$\{f(1), f(2), \ldots, f(n), \ldots\},$$

$$\{g(1), g(2), \ldots, g(n), \ldots\},$$

where

$$f(1) < f(2) < \cdots < f(n) < \cdots,$$

$$g(1) < g(2) < \cdots < g(n) < \cdots,$$

and

$$g(n) = f(f(n)) + 1$$

for all $n \geq 1$. Determine $f(240)$.

2. (IMO 1981) The function $f(x, y)$ satisfies

$$\begin{aligned} f(0, y) &= y+1, && (1) \\ f(x+1, 0) &= f(x, 1), && (2) \\ f(x+1, y+1) &= f(x, f(x+1, y)), && (3) \end{aligned}$$

for all non-negative integers x, y. Determine $f(4, 1981)$.

3. (IMO 1982) The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.