Continuous functional equations

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I'm not really an expert on this sort of thing, so let's jump straight to the

Examples

1. (Putnam 1971) Determine all polynomials $P(x)$ such that $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$.

Just like last week, you should aim to calculate as many values of P as possible, starting with $P(1)$. It shouldn't take you long to spot a pattern, and hence an example of a polynomial satisfying the equation. How do you prove that this really is the only example?

2. (Putnam 1971) Let $F(x)$ be a real valued function defined for all real x except for $x = 0$ and $x = 1$ and satisfying the functional equation

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find all functions $F(x)$ satisfying these conditions.

As before, our aim should be to calculate values of F. Bear in mind that F is unlikely to be a polynomial. Unfortunately, we can't work out any values of F directly: there is no real x for which $x = \frac{x-1}{x}$ (why not?). We're already told that F isn't defined at 0 or 1, so let's try setting $x = 2$. We obtain $F(2) + F(1/2) = 3$. Setting $x = 1/2$, we see that $F(1/2) + F(-1) = -1/2$. We seem to be getting nowhere, but setting $x = -1$ (which would also have been a good starting point) shows us that $F(2) + F(-1) = 0$. But now we have *three equations in three unknowns*, which we can solve – it turns out that $F(2) = 3/4$.

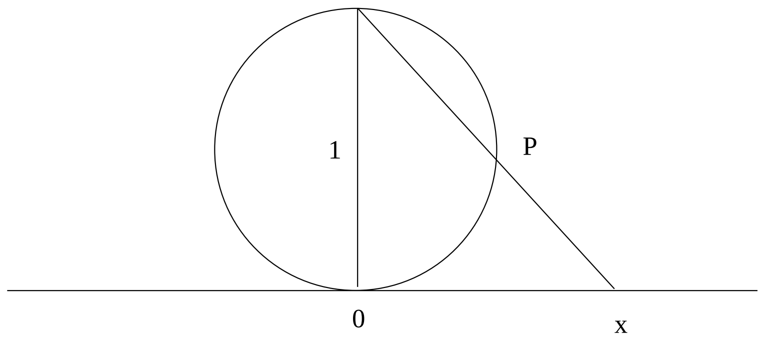

Figure 1: Stereographic projection

At this stage it is tempting to work out more values, and indeed it is possible to guess a formula from $F(3) = 17/12, F(4) = 47/24$ and $F(5) = 99/40$. However, it is simpler to observe that the same method which gave us $F(2)$ will also give us $F(x)$ for any x (except 0 and 1). Setting $g(x) = 1 - 1/x$ and using $2 \to 1/2$ to denote $g(2) = 1/2$, we see that

$$x \to 1 - \frac{1}{x} \to \frac{1}{1-x} \to x$$

so that $g(g(g(x))) = x$. Therefore, whatever $x \neq 0, 1$ we start with, we will always get three equations in the three "unknowns" $F(x), F(g(x))$ and $F(g(g(x)))$. You should be able to solve these equations to get a formula for $F(x)$.

It is interesting that $g(g(g(x))) = x$ for all $x \neq 0, 1$. A similar function with a similar property is $h(x) = \frac{1+x}{1-x}$, which satisfies $h(h(h(h(x)))) = x$. There is a pretty non-algebraic proof of this – map a point $x \in \mathbb{R}$ to a point P on a unit-diameter circle by *stereographic projection* (see Figure 1). Now mapping x to $h(x)$ corresponds to rotating the circle by ninety degrees. (Exercise: work out the details.)

3. (Putnam 1990) Find all real-valued continuously differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right)\, dt + 1990.$$

What is $f(0)$? Now, is there anything else we can observe (almost) *immediately*, without doing any detailed calculations?

4. (Putnam 1991) Suppose f and g are nonconstant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$

$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

These sorts of problems are like detective stories, where the detective is *you*. You should have a hunch as to the identity of the killers $f(x)$ and $g(x)$, but how are you going to *prove* it? You have all the evidence you need in order to force a full confession.

Homework

1. (Putnam 2000) Let $f(x)$ be a continuous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$. [**Hint.** Find some values of f. Then let $x = \cos\theta$.]

2. (College Mathematics Journal November 2008, proposed by Árpád Bényi) Call a function f *good* if $f^{(2008)}(x) = -x$ for all $x \in \mathbb{R}$, where $f^{(2008)}$ denotes the function f composed with itself 2008 times. Prove that every good function is bijective, odd, and non-monotonic. Prove also that if f is good and $x_0 \neq 0$, there exist infinitely many 5-tuples $(p_1, p_2, p_3, p_4, p_5)$ of distinct positive integers whose sum is a multiple of 5 and for which, with $q_k := f^{(p_k)}(x_0)$, $q_1 \neq q_i$ for $i = 2, 3, 4, 5$ and $q_i \neq q_{i+1}$ for $i = 2, 3, 4$.