Quadratics

$1 \lhd S_2$

$$(\alpha - \beta)^2 = (\beta - \alpha)^2 = s_1^2 - 4s_2$$

$$\alpha = \frac{1}{2}(s_1 + (\alpha - \beta))$$
$$\beta = \frac{1}{2}(s_1 + (\beta - \alpha))$$

Cubics

$A_3 \lhd S_3$

$$((\alpha-\beta)(\alpha-\gamma)(\beta-\gamma))^2 = s_1^2 s_2^2 - 4s_1^3 s_3 - 4s_2^3 - 27s_3^2 + 18s_1 s_2 s_3$$

$$t_1 = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha = \frac{1}{2}(s_1 s_2 - 3s_3 + (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma))$$
$$t_2 = \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 = \frac{1}{2}(s_1 s_2 - 3s_3 - (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma))$$

$1 \lhd A_3$

$$(\alpha+\omega\beta+\omega^2\gamma)^3 = (\beta+\omega\gamma+\omega^2\alpha)^3 = (\gamma+\omega\alpha+\omega^2\beta)^3 = s_1^3 - 3s_1 s_2 + 9s_3 + 3\omega t_1 + 3\omega^2 t_2$$
$$(\alpha+\omega\gamma+\omega^2\beta)^3 = (\beta+\omega\alpha+\omega^2\gamma)^3 = (\gamma+\omega\beta+\omega^2\alpha)^3 = s_1^3 - 3s_1 s_2 + 9s_3 + 3\omega t_2 + 3\omega^2 t_1$$

$$(\alpha+\omega\beta+\omega^2\gamma)(\alpha+\omega\gamma+\omega^2\beta) = s_1^2 - 3s_2$$
$$(\beta+\omega\gamma+\omega^2\alpha)(\beta+\omega\alpha+\omega^2\gamma) = s_1^2 - 3s_2$$
$$(\gamma+\omega\alpha+\omega^2\beta)(\gamma+\omega\beta+\omega^2\alpha) = s_1^2 - 3s_2$$

$$\alpha = \frac{1}{3}(s_1 + (\alpha+\omega\beta+\omega^2\gamma) + (\alpha+\omega\gamma+\omega^2\beta))$$
$$\beta = \frac{1}{3}(s_1 + (\beta+\omega\gamma+\omega^2\alpha) + (\beta+\omega\alpha+\omega^2\gamma))$$
$$\gamma = \frac{1}{3}(s_1 + (\gamma+\omega\alpha+\omega^2\beta) + (\gamma+\omega\beta+\omega^2\alpha))$$

Quartics

$A_4 \lhd S_4$

$$\begin{aligned}
&((\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta))^2\\
=&((\alpha\beta+\gamma\delta)-(\alpha\gamma+\beta\delta))^2\cdot((\alpha\beta+\gamma\delta)-(\alpha\delta+\beta\gamma))^2\cdot((\alpha\gamma+\beta\delta)-(\alpha\delta+\beta\gamma))^2\\
=&-128s_2^2s_4^2-4s_1^3s_3^3+16s_2^4s_4-4s_2^3s_3^2-27s_1^4s_4^2+18s_1s_2s_3^3+144s_1^2s_2s_4^2-192s_1s_3s_4^2\\
&+s_1^2s_2^2s_3^2-4s_1^2s_2^3s_4-6s_1^2s_3^2s_4+144s_2s_3^2s_4+256s_4^3-27s_3^4-80s_1s_2^2s_3s_4+18s_1^3s_2s_3s_4
\end{aligned}$$

$$\begin{aligned}
t_1&=(\alpha\beta+\gamma\delta)^2(\alpha\gamma+\beta\delta)+(\alpha\gamma+\beta\delta)^2(\alpha\delta+\beta\gamma)+(\alpha\delta+\beta\gamma)^2(\alpha\beta+\gamma\delta)\\
&=\frac{1}{2}(s_1s_2s_3+8s_2s_4-3s_1^2s_4-3s_3^2+(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta))\\
t_2&=(\alpha\beta+\gamma\delta)(\alpha\gamma+\beta\delta)^2+(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)^2+(\alpha\delta+\beta\gamma)(\alpha\beta+\gamma\delta)^2\\
&=\frac{1}{2}(s_1s_2s_3+8s_2s_4-3s_1^2s_4-3s_3^2-(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta))
\end{aligned}$$

$V \lhd A_4$

$$\begin{aligned}
((\alpha\beta+\gamma\delta)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\delta+\beta\gamma))^3&=((\alpha\gamma+\beta\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\beta+\gamma\delta))^3\\
&=((\alpha\delta+\beta\gamma)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\gamma+\beta\delta))^3\\
&=s_2^3+9s_3^2+9s_1^2s_4-24s_2s_4-3s_1s_2s_3+3\omega t_1+3\omega^2t_2\\
((\alpha\beta+\gamma\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\gamma+\beta\delta))^3&=((\alpha\gamma+\beta\delta)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\delta+\beta\gamma))^3\\
&=((\alpha\delta+\beta\gamma)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\beta+\gamma\delta))^3\\
&=s_2^3+9s_3^2+9s_1^2s_4-24s_2s_4-3s_1s_2s_3+3\omega t_2+3\omega^2t_1
\end{aligned}$$

$$\begin{aligned}
((\alpha\beta+\gamma\delta)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\delta+\beta\gamma))\cdot((\alpha\beta+\gamma\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\gamma+\beta\delta))&=s_2^2+12s_4-3s_1s_3\\
((\alpha\gamma+\beta\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\beta+\gamma\delta))\cdot((\alpha\gamma+\beta\delta)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\delta+\beta\gamma))&=s_2^2+12s_4-3s_1s_3\\
((\alpha\delta+\beta\gamma)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\gamma+\beta\delta))\cdot((\alpha\delta+\beta\gamma)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\beta+\gamma\delta))&=s_2^2+12s_4-3s_1s_3
\end{aligned}$$

$$\begin{aligned}
p_1=\alpha\beta+\gamma\delta&=\frac{1}{3}(s_2+((\alpha\beta+\gamma\delta)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\delta+\beta\gamma))+((\alpha\beta+\gamma\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\gamma+\beta\delta)))\\
p_2=\alpha\gamma+\beta\delta&=\frac{1}{3}(s_2+((\alpha\gamma+\beta\delta)+\omega(\alpha\delta+\beta\gamma)+\omega^2(\alpha\beta+\gamma\delta))+((\alpha\gamma+\beta\delta)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\delta+\beta\gamma)))\\
p_3=\alpha\delta+\beta\gamma&=\frac{1}{3}(s_2+((\alpha\delta+\beta\gamma)+\omega(\alpha\beta+\gamma\delta)+\omega^2(\alpha\gamma+\beta\delta))+((\alpha\delta+\beta\gamma)+\omega(\alpha\gamma+\beta\delta)+\omega^2(\alpha\beta+\gamma\delta)))
\end{aligned}$$

$1 \triangleleft V$

$$\begin{aligned}
(\alpha+\beta-(\gamma+\delta))^2 = (\gamma+\delta-(\alpha+\beta))^2 = s_1^2 - 4s_2 + 4p_1 \\
(\alpha+\gamma-(\beta+\delta))^2 = (\beta+\delta-(\alpha+\gamma))^2 = s_1^2 - 4s_2 + 4p_2 \\
(\alpha+\delta-(\beta+\gamma))^2 = (\beta+\gamma-(\alpha+\delta))^2 = s_1^2 - 4s_2 + 4p_3
\end{aligned}$$

$$(\alpha+\beta-(\gamma+\delta)) \cdot (\alpha+\gamma-(\beta+\delta)) \cdot (\alpha+\delta-(\beta+\gamma)) = s_1^3 - 4s_1s_2 + 8s_3$$

$$\begin{aligned}
\alpha &= \frac{1}{4}(s_1 + (\alpha+\beta-(\gamma+\delta)) + (\alpha+\gamma-(\beta+\delta)) + (\alpha+\delta-(\beta+\gamma))) \\
\beta &= \frac{1}{4}(s_1 + (\alpha+\beta-(\gamma+\delta)) + (\beta+\gamma-(\alpha+\delta)) + (\beta+\delta-(\alpha+\gamma))) \\
\gamma &= \frac{1}{4}(s_1 + (\alpha+\gamma-(\beta+\delta)) + (\beta+\gamma-(\alpha+\delta)) + (\gamma+\delta-(\alpha+\beta))) \\
\delta &= \frac{1}{4}(s_1 + (\alpha+\delta-(\beta+\gamma)) + (\beta+\delta-(\alpha+\gamma)) + (\gamma+\delta-(\alpha+\beta)))
\end{aligned}$$