Rainbow Turán problems

Amites Sarkar

Western Washington University

2019 Coast Combinatorics Conference

Joint work with Dan Johnston (Grand Valley State University)
and Cory Palmer (University of Montana)

Recall that the Turán number $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph on n vertices.

Turán's theorem:

$$\mathrm{ex}(n, K_r) = \left(1 - \frac{1}{r-1} + o(1)\right) \frac{n^2}{2}$$

Erdős-Stone theorem:

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right) \frac{n^2}{2}$$

where $\chi(F)$ is the chromatic number of F.

When F is bipartite the behavior of $\mathrm{ex}(n, F)$ is not always known.

The rainbow Turán number $\mathrm{ex}^*(n, F)$ is the maximum number of edges in an n-vertex graph that has a proper edge-coloring with no rainbow copy of F (i.e. in which all the edges of F get different colors).

Introduced by Keevash, Mubayi, Sudakov and Verstraëte in 2007.

How do we get bounds on $\mathrm{ex}^*(n, F)$?

Lower bound: Construct an n-vertex graph G with a proper edge-coloring without a rainbow copy of F.

Upper bound: Show that every proper edge-coloring of every n-vertex graph G with enough edges contains a rainbow copy of F.

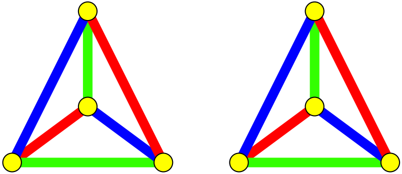
Warmup: what is the relationship between $\mathrm{ex}(n, F)$ and $\mathrm{ex}^*(n, F)$?

$\mathrm{ex}(n, F) \leq \mathrm{ex}^*(n, F)$

$\mathrm{ex}(n, K_3) = \mathrm{ex}^*(n, K_3)$

$\mathrm{ex}(n, P_3) < \mathrm{ex}^*(n, P_3)$

$\mathrm{ex}(n, P_3) < \mathrm{ex}^*(n, P_3)$



Theorem (Keevash, Mubayi, Sudakov and Verstraëte 2007)

If F has chromatic number $\chi(F) > 2$, then

$$\mathrm{ex}^*(n, F) = (1 + o(1))\mathrm{ex}(n, F).$$

Idea of proof: Given a proper edge-coloring of an n-vertex graph G with $(1 + o(1))\mathrm{ex}(n, F)$ edges, find a large complete $\chi(F)$-partite graph H in G, and then greedily construct a rainbow copy of F inside H.

Theorem (Keevash, Mubayi, Sudakov and Verstraëte 2007)

$$\mathrm{ex}^*(n, K_{s,t}) = O(n^{2-1/s})$$

$\mathrm{ex}^*(n, C_{2k})$ is related to B_k^*-sets in additive number theory.

Definition

A subset A of an abelian group G is a B_k^*-set if A does not contain disjoint k-subsets B and C with the same sum.

Given a B_k^*-set A, construct a properly edge-colored bipartite graph $G = (X, Y)$ as follows. X and Y are both copies of G. Given $x \in X$ and $y \in Y$, if $x - y \in A$, draw edge xy and color it $x - y$.

G does not contain a rainbow C_{2k}.

Theorem (Bose and Chowla 1960)

$G = \mathbb{Z}/n\mathbb{Z}$ contains a B_k^*-set of size $(1 + o(1))n^{1/k}$.

Consequently, $\mathrm{ex}^*(n, C_{2k}) = \Omega(n^{1+1/k})$. An upper bound on $\mathrm{ex}^*(n, C_{2k})$ would yield a purely combinatorial upper bound for the maximum size of a B_k^*-set.

Theorem (Keevash, Mubayi, Sudakov and Verstraëte 2007)

$$\mathrm{ex}^*(n, C_4) = \Theta(n^{3/2})$$

$$\mathrm{ex}^*(n, C_6) = \Theta(n^{4/3})$$

Theorem (Das, Lee and Sudakov 2012)

$$\mathrm{ex}^*(n, C_{2k}) = O\left(n^{1+\frac{(1+\epsilon_k)\ln k}{k}}\right)$$

Theorem (Ruzsa 1993)

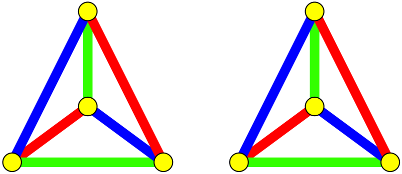
A B_k^*-set on $\{1, 2, \ldots, n\}$ has at most $(1+o(1))k^{2-1/k}n^{1/k}$ elements.

Write P_ℓ for the path with ℓ edges.

Conjecture (Keevash, Mubayi, Sudakov and Verstraëte)

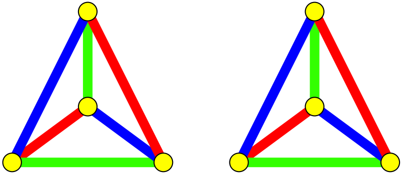
$$\frac{(\ell-1)n}{2} \sim \mathrm{ex}(n, P_\ell) \leq \mathrm{ex}^*(n, P_\ell) \sim \frac{(f(\ell)-1)n}{2},$$

where $f(\ell)$ is maximal such that a proper edge-coloring of $K_{f(\ell)}$ does not contain a rainbow P_ℓ.



Observation (Keevash, Mubayi, Sudakov and Verstraëte)

$$\mathrm{ex}^*(n, P_3) = \frac{3n}{2} + O(1)$$



$f(3) = 4$ $\quad$ $f(4) = 4$ $\quad$ $f(5) = 6$

KMSV Conjecture ($\ell = 4$)

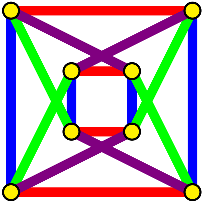
$$\mathrm{ex}^*(n, P_4) = \frac{3n}{2} + O(1),$$

Proposition (Johnston, Palmer and Sarkar 2017)

$$\mathrm{ex}^*(n, P_4) = 2n + O(1)$$

Consequently, the conjecture is false when $l = 4$.

Lower bound comes from disjoint copies of the following graph:



Upper bound on $\mathrm{ex}^*(n, P_4)$ is case analysis.

Conjecture (Keevash, Mubayi, Sudakov and Verstraëte)

$$\mathrm{ex}^*(n, P_\ell) \sim \frac{(f(\ell)-1)n}{2},$$

where $f(\ell)$ is maximal such that a proper edge-coloring of $K_{f(\ell)}$ does not contain a rainbow P_ℓ.

Proposition (Johnston and Rombach 2019+)

$$\mathrm{ex}^*(n, P_\ell) \geq \frac{\ell n}{2} + O(1),$$

Conjecture (Andersen 1989)

$$f(\ell) \leq \ell + 1$$

Theorem (Alon, Pokrovskiy and Sudakov 2016)

$$f(\ell) \leq \ell + O(\ell^{3/4})$$

Theorem (Johnston, Palmer and Sarkar 2017)

$$\mathrm{ex}^*(n, P_\ell) \leq \left\lceil \frac{3\ell - 2}{2} \right\rceil n.$$

Idea of proof: If G has average degree 3ℓ, it contains a subgraph H of minimum degree $3\ell/2$. By a theorem of Babu, Chandran and Rajendraprasad, a proper coloring of H contains a rainbow P_ℓ.

Theorem (Ergemlidze, Győri and Methuku 2018+)

$$\mathrm{ex}^*(n, P_\ell) < \frac{(9\ell + 5)n}{7}$$

Theorem (Lidický, Liu and Palmer 2013)

Let F be a forest of k stars $S_1, S_2, \ldots, S_k$, such that $e(S_j) \leq e(S_{j+1})$ for each j. Then

$$\mathrm{ex}(n, F) = \max_{0 \leq i \leq k-1} \left\{ i(n-i) + \binom{i}{2} + \left\lfloor \frac{(e(S_{k-i}) - 1)(n-i)}{2} \right\rfloor \right\}.$$

Theorem (Johnston, Palmer and Sarkar 2017)

Let F be a forest of k stars. Suppose that G is an edge-maximal properly edge-colored graph on n vertices containing no rainbow copy of F. Then, for n large enough, either G is an edge-maximal $(e(F) - 1)$-edge-colorable graph, or G is a set of $k - 1$ universal vertices connected to an independent set of size $n - k + 1$.

Theorem (Johnston, Palmer and Sarkar 2017)

Let F be a forest of k stars. Suppose that G is an edge-maximal properly edge-colored graph on n vertices containing no rainbow copy of F. Then, for n large enough, either G is an edge-maximal $(e(F)-1)$-edge-colorable graph, or G is a set of $k-1$ universal vertices connected to an independent set of size $n-k+1$.

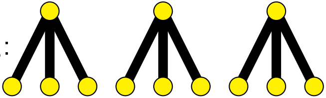
Two options for the lower bound construction:

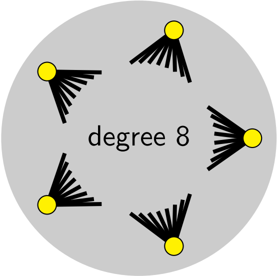
- $(e(F)-1)$-edge-colorable graph (not enough colors)

- $k-1$ universal vertices connected to an independent set (no copy of F)

Consider F to be 3 stars each of size 3:

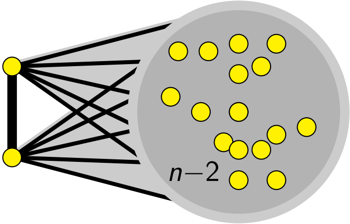
$3K_{1,3}$:





8-edge-colorable

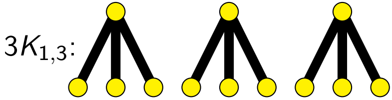
$\sim 4n$ edges

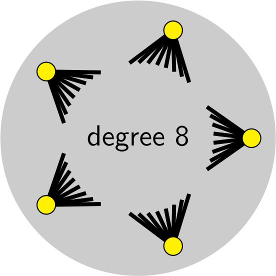


2 universal vertices

$\sim 2n$ edges

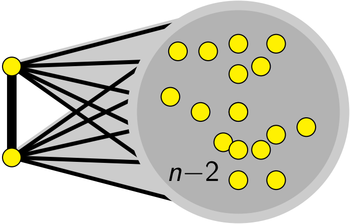
Consider F to be 3 stars each of size 3:





8-edge-colorable

$\mathrm{ex}^*(n, F) \sim 4n$



2 universal vertices + cycle

$\mathrm{ex}(n, F) \sim 3n$

Theorem (Johnston, Palmer and Sarkar 2017)

Let F be a forest of k stars. Suppose that G is an edge-maximal properly edge-colored graph on n vertices containing no rainbow copy of F. Then, for n large enough, either G is an edge-maximal $(e(F)-1)$-edge-colorable graph, or G is a set of $k-1$ universal vertices connected to an independent set of size $n-k+1$.

Corollary

Let F be a matching of size k. Then for sufficiently large n

$$\mathrm{ex}^*(n, F) = \mathrm{ex}(n, F) = \binom{k-1}{2} + (k-1)(n-k+1).$$

Some open problems:

- Improve the bounds on $\mathrm{ex}^*(n, C_{2k})$.

- Improve the bounds on $\mathrm{ex}^*(n, P_\ell)$.

- How many edges force a rainbow cycle of ANY length? We know (from Das, Lee, Sudakov):

$$n \ln n \leq f(n) \leq n^{1+\epsilon}.$$

Thank you for your attention!